Claim: If $y = \phi_1(t)$ and $y = \phi_2(t)$ are linearly independent solutions to $y'' + py' + qy = 0$, then

$$\boxed{\text{general solution is } y(t) = c_1\phi_1(t) + c_2\phi_2(t)}$$

That is any solution to this linear 2nd order homogeneous differential equation can be written as a linear combination of the linear independent functions $y = \phi_1(t)$ and $y = \phi_2(t)$.

Thus for a 2^{nd} order linear homogeneous differential equation,

we need to find 2 linearly independent solutions

in order to find the general solution

Section 3.4: If $b^2 - 4ac = 0$, then $r_1 = r_2$. repeated root

Hence one sol'n is $y = e^{r_1 t}$ Need 2^{nd} sol'n to $ay'' + by' + cy = 0$.

If $y = e^{rt}$ is a solution, $y = ce^{rt}$ is a solution.

How about $y = v(t)e^{rt}$?

$$y' = v'(t)e^{rt} + v(t)re^{rt}$$

$$y'' = v''(t)e^{rt} + v'(t)re^{rt} + v'(t)re^{rt} + v(t)r^2e^{rt}$$

$$= v''(t)e^{rt} + 2v'(t)re^{rt} + v(t)r^2e^{rt}$$

$$a(v''e^{rt} + 2v're^{rt} + vr^2e^{rt}) + b(v'e^{rt} + vre^{rt}) + cve^{rt} = 0$$

$$a(v''(t) + 2v'(t)r + v(t)r^2) + b(v'(t) + v(t)r) + cv(t) = 0$$

$$av''(t) + 2av'(t)r + av(t)r^2 + bv'(t) + bv(t)r + cv(t) = 0$$

$$av''(t) + (2ar + b)v'(t) + (ar^2 + br + c)v(t) = 0$$

$$av''(t) + (2ar + b)v'(t) + (ar^2 + br + c)v(t) = 0$$

$$av''(t) + (2a(\frac{-b}{2a}) + b)v'(t) + 0 = 0$$

since $ar^2 + br + c = 0$ and $r = \frac{-b}{2a}$

$av''(t) + (-b + b)v'(t) = 0.$ Thus $av''(t) = 0.$

Hence $v''(t) = 0$ and $v'(t) = k_1$ and $v(t) = k_1 t + k_2$

Hence $v(t)e^{r_1 t} = (k_1 t + k_2)e^{r_1 t}$ is a soln

Thus $te^{r_1 t}$ is a nice second solution.

Hence general solution is $y = c_1 e^{r_1 t} + c_2 t e^{r_1 t}$

Section 3.4: Reduction of order

2nd order transform it into a first order DE

Suppose $y = \phi_1(t)$ is a solution to $y'' + p(t)y' + q(t)y = 0$

ϕ_1 is homog soln

Guess $\boxed{y = \underbrace{v(t)}_{\text{unknown}}\underbrace{\phi_1}_{\text{homog}}(t)}$ is also a solution. ⇒ when plug in $y = v\phi$, lots of stuff will cancel out

Solve for unknown function $v(t)$ by plugging in:

$y = v\phi_1$

$y' = v\phi_1' + v'\phi_1$

$y'' = (v\phi_1'' + v'\phi_1') + (v'\phi_1' + v''\phi_1)$

$= v\phi_1'' + 2v'\phi_1' + v''\phi_1$

(leaving us w/ a DE that we can solve)

$y = \phi_1(t)$ is a solution to $y'' + p(t)y' + q(t)y = 0$

implies $\phi_1'' + p(t)\phi_1' + q(t)\phi_1 = 0$

$$y = v(t)\phi_1(t) \implies y' = v'(t)\phi_1(t) + v(t)\phi_1'(t)$$

$$y'' = v''(t)\phi_1(t) + v'(t)\phi_1'(t) + v'(t)\phi_1'(t) + v(t)\phi_1''(t)$$

$$= v''(t)\phi_1(t) + 2v'(t)\phi_1'(t) + v(t)\phi_1''(t)$$

$$(v''\phi_1 + 2v'\phi_1 + v\phi_1'') + p(v'\phi_1 + v\phi_1') + qv\phi = 0$$

$$y'' + p(t)y' + q(t)y = 0$$

$$\phi_1'' + p\phi_1' + q\phi = 0$$

$$v''(t)\phi_1(t) + 2v'(t)\phi_1'(t) + v(t)\phi_1''(t) \qquad y''$$

$$+ p(t)[v'(t)\phi_1(t) + v(t)\phi_1'(t)] \qquad py'$$

$$+ q(t)[v(t)\phi_1(t)] = 0 \qquad qy$$

$+$

$$v''\phi_1 + 2v'\phi_1' + pv'\phi_1 + v\phi_1'' + pv\phi_1' + qv\phi_1$$

$$= v''\phi_1 + 2v'\phi_1' + pv'\phi_1 + v\,\underbrace{\cancel{(\phi_1'' + p\phi_1' + q\phi_1)}}_{0}$$

$$\Rightarrow v''\phi_1 + v'(2\phi_1' + \phi_1) = 0$$

Section 3.4: Reduction of order

Suppose $y = \phi_1(t)$ is a solution to $\overset{\phi_1}{\cancel{y}}'' + p(t)\overset{\phi_1}{\cancel{y}}' + q(t)\overset{\phi_1}{\cancel{y}} = 0$

Guess $y = v(t)\phi_1(t)$ is also a solution.

Solve for unknown function $v(t)$ by plugging in:

simplication

$$v''(t)\phi_1(t) + 2v'(t)\phi_1'(t) + p(t)v'(t)\phi_1(t) = 0$$

$$v''\phi_1 + v'\left(2\phi_1'(t) + p\,\phi_1(t)\right) = 0$$

Reduction of order: Let $w = v' \Rightarrow w' = v''$

$$w'\phi_1 + w\left(2\phi_1' + p\,\phi_1\right) = 0 \Leftarrow$$ 1rst order DE

2nd order $\to$ 1rst order

Section 3.4: Reduction of order

Suppose $y = \phi_1(t)$ is a solution to $y'' + p(t)y' + q(t)y = 0$

Guess $y = v(t)\phi_1(t)$ is also a solution.

Solve for unknown function $v(t)$ by plugging in:

$$v''(t)\phi_1(t) \; + \; v'(t)[2\phi_1'(t) \; + \; p(t)\phi_1(t)] = 0$$

$$v''(t)\phi_1(t) \ + \ v'(t)[2\phi_1'(t) \ + \ p(t)\phi_1(t)] = 0$$

2nd order

$$v''(t)\phi_1(t) + v'(t)[2\phi_1'(t) + p(t)\phi_1(t)] = 0$$

Let $w(t) = v'(t)$, then $w'(t) = v''(t)$

$\boxed{w' = \dfrac{dw}{dt}}$

← reduction of order

first order

$$w'(t)\phi_1(t) + w(t)[2\phi_1'(t) + p(t)\phi_1(t)] = 0$$

linear and seperable

$$\frac{dw}{\cancel{dt}}\cancel{dt}\,\phi_1(t) + w\,[2\phi_1'(t) + p(t)\phi_1(t)]\,dt = 0\,dt$$

separate variables

$$\frac{dw\,\phi_1(t)}{w\,\phi_1(t)} = \frac{-\cancel{w}\,dt}{\cancel{w}}\,\frac{[2\phi_1'(t) + p(t)\phi_1(t)]}{\phi_1(t)}$$

$$v''(t)\phi_1(t) \;+\; v'(t)[2\phi_1'(t) \;+\; p(t)\phi_1(t)] = 0$$

Let $w(t) = v'(t)$, then $w'(t) = v''(t)$

$$w'(t)\phi_1(t) \;+\; w(t)[2\phi_1'(t) \;+\; p(t)\phi_1(t)] = 0$$

$$w'(t)\phi_1(t) \;=\; -w(t)[2\phi_1'(t) \;+\; p(t)\phi_1(t)]$$

$w = \frac{dw}{dt}$

$$\frac{w'(t)}{w(t)} = \frac{2\phi_1'(t) \;+\; p(t)\phi_1(t)}{\phi_1(t)}$$

$$\cancel{dt}\,\frac{dw}{\cancel{dt}}\left(\frac{1}{w}\right) = \int \frac{w'(t)}{w(t)} = \int \frac{2\phi_1'(t) \;+\; p(t)\phi_1(t)}{\phi_1(t)}\,dt$$

$$\int \frac{dw}{w} = \int \frac{2\phi_1'(t) \;+\; p(t)\phi_1(t)}{\phi_1(t)}\,dt$$

$$e^{\ln|w|} = e^{\int \sim}$$

$$w = Ce^{A(t)}$$

$$\frac{w'}{w}$$

$\ln|w|$

$$w = Ce^{\sim}$$

$1r^2 + \frac{b}{a} r + \frac{c}{a} = 0$

$$\frac{w'(t)}{w(t)} = \frac{2\phi_1'(t) + p(t)\phi_1(t)}{\phi_1(t)}$$

$ar^2 + br + c = 0$

$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, $b^2 - 4ac = 0$

$$\boxed{\frac{dw}{w} = \frac{2\phi_1'(t) + p(t)\phi_1(t)}{\phi_1(t)}dt}$$

Example: $ay'' + by' + cy = 0$, $\phi_1(t) = e^{rt}$, $p(t) = \frac{b}{a}$

repeated root case

$$\frac{dw}{w} = \frac{2r\cancel{e^{rt}} + \frac{b}{a}\cancel{e^{rt}}}{\cancel{e^{rt}}}$$

where $r = \frac{-b}{2a}$

etc . . . . .

$$\boxed{w(t) = v'(t)}$$

$v = $ ~

If $b^2 - 4ac > 0$, general sol'n is $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$.

If $b^2 - 4ac < 0$, change format to linear combination of real-valued functions instead of complex valued functions by using Euler's formula.

general solution is $y = c_1 e^{dt} cos(nt) + c_2 e^{dt} sin(nt)$
where $r = d \pm in$

If $b^2 - 4ac = 0$, $r_1 = r_2$, so need 2nd (independent) solution: $te^{r_1 t}$

Hence general solution is $y = c_1 e^{r_1 t} + c_2 t e^{r_1 t}$.

Initial value problem: use $y(t_0) = y_0$, $y'(t_0) = y'_0$ to solve for c_1, c_2 to find unique solution.

why

assuming constant coefficients

$ay'' + by' + cy = 0$

Solve: $\boxed{y'' + y = 0}$, $y(0) = -1$, $y'(0) = -3$

Guess $y = e^{rt}$, plug in & solve for r.

$$r^2 + 1 = 0 \Rightarrow r^2 = -1 \Rightarrow r = \pm\sqrt{-1} = \pm i$$

general soln $=$ $\underline{y = c_1 \cos t + c_2 \sin(t)}$

IVP

$y(0) = -1$: $y = c_1 \cos t + c_2 \sin t \Rightarrow -1 = c_1(1) + c_2(0)$

$\Rightarrow c_1 = -1$

$y'(0) = -3$: $y' = -c_1 \sin t + c_2 \cos t \Rightarrow -3 = -c_1(0) + c_2(1)$

$\Rightarrow c_2 = -3$

$\boxed{\text{IVP soln: } y = -\cos t - 3\sin t}$

Solve: $y'' + y = 0$, $y(0) = -1$, $y'(0) = -3$

non simplified general sol'n

$r^2 + 1 = 0$ implies $r^2 = -1$. Thus $r = \pm i$.

Not ACCEPTABLE since not simplified

NOT RECOMMENDED: work with $\boxed{y = c_1 e^{it} + c_2 e^{-it}}$

$$y' = ic_1 e^{it} - ic_2 e^{-it}$$

$y(0) = -1$: $\quad -1 = c_1 e^0 + c_2 e^0$ implies $-1 = c_1 + c_2$.

$y'(0) = -3$: $\quad -3 = ic_1 e^0 - ic_2 e^0$ implies $-3 = ic_1 - ic_2$.

$$-1i = ic_1 + ic_2.$$

$$-3 = ic_1 - ic_2.$$

$$-1i = ic_1 + ic_2.$$

$$-3 = ic_1 - ic_2.$$

since these eqns are from real valued IVP

$2ic_1 = -3 - i$ implies $c_1 = \frac{-3i-i^2}{-2} = \frac{3i-1}{2}$

$2ic_2 = 3 - i$ implies $c_2 = \frac{3i-i^2}{-2} = \frac{-3i-1}{2}$

Note these are complex conjugates

Euler's formula: $e^{ix} = cos(x) + isin(x)$

$$y = (\frac{3i-1}{2})e^{it} + (\frac{-3i-1}{2})e^{-it}$$

NOT simplified real valued function

$$= (\frac{3i-1}{2})[cos(t) + isin(t)] + (\frac{-3i-1}{2})[cos(-t) + isin(-t)]$$

$$= (\frac{3i-1}{2})[cos(t) + isin(t)] + (\frac{-3i-1}{2})[cos(t) - isin(t)]$$

FOIL

$$= (\frac{3i}{2})cos(t) + (\frac{3i}{2})isin(t) + (\frac{-1}{2})cos(t) + (\frac{-1}{2})isin(t)$$
$$+(\frac{-3i}{2})cos(t) - (\frac{-3i}{2})isin(t) + (\frac{-1}{2})cos(t) - (\frac{-1}{2})isin(t)$$

$$= (\frac{3i}{2})isin(t) + (\frac{-1}{2})cos(t) + (\frac{3i}{2})isin(t) + (\frac{-1}{2})cos(t)$$

$$= -(\frac{3}{2})sin(t) - (\frac{1}{2})cos(t) - (\frac{3}{2})sin(t) - (\frac{1}{2})cos(t)$$

$$= -3sin(t) - 1cos(t)$$

SIMPLIFIED ANSWER

Solve: $y'' + y = 0$, $y(0) = -1$, $y'(0) = -3$

$r^2 + 1 = 0$ implies $r^2 = -1$. Thus $r = \pm i$.

RECOMMENDED Method:

Since $r = 0 \pm 1i$, $\boxed{y = c_1 cos(t) + c_2 sin(t)}$ general soln

Then $y' = -c_1 sin(t) + c_2 cos(t)$

$y(0) = -1$: $\quad -1 = c_1 cos(0) + c_2 sin(0)$ implies $-1 = c_1$

$y'(0) = -3$: $\quad -3 = -c_1 sin(0) + c_2 cos(0)$ implies $-3 = c_2$

NOTE UNIQUE SOLN for c_1 & c_2

unique c_1 & c_2 $\Rightarrow$ unique soln

soln to IVP

Thus IVP solution: $\boxed{y = -cos(t) - 3sin(t)}$

When does the following IVP have unique sol'n: 3.2

IVP: $a(t)y'' + b(t)y' + c(t)y = 0$, $y(t_0) = y_0$, $y'(t_0) = y_1$.

Don't need to assume a, b, c are constants in sectn 3.2

Suppose $y = c_1\phi_1(t) + c_2\phi_2(t)$ is a solution to $ay'' + by' + cy = 0$.

Assuming general soln exists

Then $y' = c_1\phi_1'(t) + c_2\phi_2'(t)$

$y(t_0) = y_0$: $y_0 = c_1\phi_1(t_0) + c_2\phi_2(t_0)$

$y'(t_0) = y_1$: $y_1 = c_1\phi_1'(t_0) + c_2\phi_2'(t_0)$

To find IVP solution, need to solve above system of two equations for the unknowns c_1 and c_2.

sol'n exists and is unique for c_1 & c_2

Note the IVP has a unique solution if and only if the above system of two equations has a unique solution for c_1 and c_2.

Soln to IVP exists & is unique if

$$y(t_0) = y_0: \quad y_0 = c_1\phi_1(t_0) + c_2\phi_2(t_0) = y_0$$
$$y'(t_0) = y_1: \quad y_1 = c_1\phi_1'(t_0) + c_2\phi_2'(t_0) = y_1$$

linear algebra ~~problem~~ has a soln which is unique for c_1 & c_2

Note that in these equations c_1 and c_2 are the unknowns and $y_0, \phi_1(t_0), \phi_2(t_0), y_1, \phi_1'(t_0), \phi_2'(t_0)$ are the constants. We can translate this linear system of equations into matrix form:

$$\begin{bmatrix} \phi_1(t_0) & \phi_2(t_0) \\ \phi_1'(t_0) & \phi_2'(t_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}$$

Note this equation has a unique solution if and only if

$$\det \begin{vmatrix} \phi_1(t_0) & \phi_2(t_0) \\ \phi_1'(t_0) & \phi_2'(t_0) \end{vmatrix} \neq 0$$

Definition: The Wronskian of two differential functions, ϕ_1 and ϕ_2 is

$$W(\phi_1, \phi_2)(t) = \phi_1(t)\phi_2'(t) - \phi_1'(t)\phi_2(t) = \begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix}(t)$$

Definition: The Wronskian of two differential functions, ϕ_1 and ϕ_2 is

$$W(\phi_1, \phi_2)(t) = \phi_1\phi_2' - \phi_1'\phi_2 = \begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix}(t)$$

$W(\phi_1, \phi_2)(t_0)$ is the determinant of the coefficient matrix when solving an IVP at $t = t_0$

Examples:

1.) $W(\cos(t), \sin(t)) = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix}$

$= \cos^2 t - (-\sin^2 t)$

$= \cos^2 t + \sin^2 t = 1 \neq 0$

$\Rightarrow$ IVP will have unique soln if general soln

$y = c_1 \cos t + c_2 \sin t$

Examples:

1.) $\mathrm{W}(cos(t),\, sin(t)) = \begin{vmatrix} cos(t) & sin(t) \\ -sin(t) & cos(t) \end{vmatrix}$

$$= cos^2(t) + sin^2(t) = 1 > 0.$$

2.) W($\underline{e^{dt}cos(nt)}$, $\underline{e^{dt}sin(nt)}$)

$$= \begin{vmatrix} e^{dt}cos(nt) & e^{dt}sin(nt) \\ de^{dt}cos(nt) - ne^{dt}sin(nt) & de^{dt}sin(nt) + ne^{dt}cos(nt) \end{vmatrix}$$

$$= e^{dt}cos(nt)(de^{dt}sin(nt) + ne^{dt}cos(nt)) - e^{dt}sin(nt)(de^{dt}cos(nt) - ne^{dt}sin(nt))$$

$$= e^{2dt}[cos(nt)(dsin(nt) + ncos(nt)) - sin(nt)(dcos(nt) - nsin(nt))]$$

$$= e^{2dt}[dcos(nt)sin(nt) + ncos^2(nt) - dsin(nt)cos(nt) + nsin^2(nt)])$$

$$= e^{2dt}[ncos^2(nt) + nsin^2(nt)] = ne^{2dt}[cos^2(nt) + sin^2(nt)]$$

if roots are complex

$\Rightarrow$ IVP has unique soln

$= ne^{2dt} \neq 0$ for all t.