Induction proof will be graded

**Summary of sections 3.1, 3, 4:**

Solve linear homogeneous 2nd order DE with constant coefficients.

Solve $ay'' + by' + cy = 0$. Educated guess $y = e^{rt}$, then

$ar^2\cancel{e^{rt}} + br\cancel{e^{rt}} + c\cancel{e^{rt}} = 0$ implies $\boxed{ar^2 + br + c = 0,}$

Suppose $r = r_1, r_2$ are solutions to $ar^2 + br + c = 0$

$$r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If $r_1 \neq r_2$, then $b^2 - 4ac \neq 0$.

Hence a general solution is $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

2 real solns

If $b^2 - 4ac > 0$, general sol'n is $\boxed{y = c_1 e^{r_1 t} + c_2 e^{r_2 t}}$.

---

2 complex solns

If $b^2 - 4ac < 0$, change format to linear combination of real-valued functions instead of complex valued functions by using Euler's formula.

general solution is $\boxed{y = c_1 e^{dt} cos(nt) + c_2 e^{dt} sin(nt)}$ where $r = d \pm in$

---

1 repeated root

If $b^2 - 4ac = 0$, $r_1 = r_2$, so need 2nd (independent) solution: $te^{r_1 t}$

Hence general solution is $\boxed{y = c_1 e^{r_1 t} + c_2 t e^{r_1 t}}$.

---

Initial value problem: use $y(t_0) = y_0$, $y'(t_0) = y'_0$ to solve for $c_1, c_2$ to find unique solution.

take derivative of soln

3.2: Theory (why everything works)

compare to ch 2

**1st order LINEAR differential equation:**

Thm 2.4.1: If $p : (a, b) \to R$ and $g : (a, b) \to R$ are continuous and $a < t_0 < b$, then there exists a unique function $y = \phi(t)$, $\phi : (a, b) \to R$ that satisfies the

$$\text{IVP: } y' + p(t)y = g(t), \quad y(t_0) = y_0$$

Proof 1: Constructive proof (use integrating factor to find solution).

Proof 2 outline: Use linearity.

need one soln

## 1st order LINEAR differential equation:

Thm 2.4.1: If $p : (a, b) \to R$ and $g : (a, b) \to R$ are continuous and $a < t_0 < b$, then there exists a unique function $y = \phi(t)$, $\phi : (a, b) \to R$ that satisfies the

$$\text{IVP: } y' + p(t)y = g(t), \quad y(t_0) = y_0$$

Thm: If $y = \phi_1(t)$ is a solution to homogeneous equation, $y' + p(t)y = 0$, then $y = c\phi_1(t)$ is the general solution to this equation.

If in addition $y = \psi(t)$ is a solution to non-homogeneous equation, $y' + p(t)y = g(t)$, then $y = c\phi_1(t) + \psi(t)$ is the general solution to this equation.

compare to solns using integrating factor

compare to constructive proof using integrating factor

Look at your 2.4 HW # 19, 20, 21

## 1st order LINEAR differential equation:

Thm 2.4.1: If $p : (a, b) \to R$ and $g : (a, b) \to R$ are continuous and $a < t_0 < b$, then
$\exists!$ function $y = \phi(t)$, $\phi : (a, b) \to R$ that satisfies

$$\text{IVP: } 1 y' + p(t)y = g(t), \quad y(t_0) = y_0$$

domain

---

## 2nd order LINEAR differential equation:

Thm 3.2.1: If $p : (a, b) \to R$, $q : (a, b) \to R$, and
$g : (a, b) \to R$ are continuous and $\boxed{a < t_0 < b}$ then
$\exists!$ function $y = \phi(t)$, $\phi : (a, b) \to R$ that satisfies

$$\text{IVP: } 1 y'' + p(t)y' + q(t)y = g(t), \quad y(t_0) = y_0, \quad y'(t_0) = y'_0$$

ch 4 : $n^{th}$ order linear

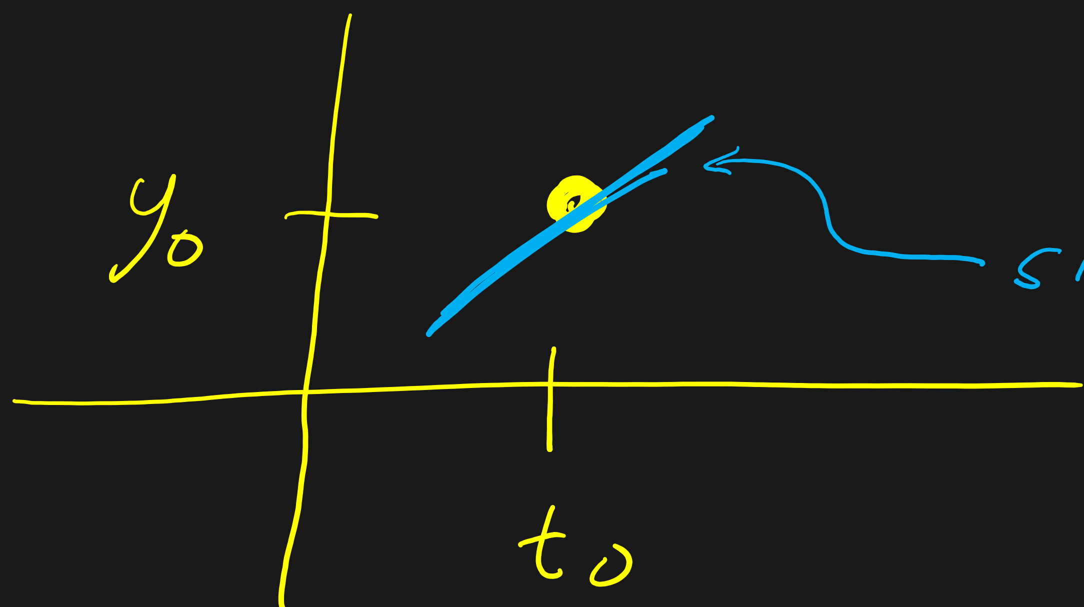
**2nd order LINEAR differential equation:**

Thm 3.2.1: If $p:(a,b)\to R$, $q:(a,b)\to R$, and $g:(a,b)\to R$ are continuous and $a<t_0<b$, then $\exists!$ function $y=\phi(t)$, $\phi:(a,b)\to R$ that satisfies

IVP: $y''+p(t)y'+q(t)y=g(t)$, $\boxed{y(t_0)=y_0,}$ $y'(t_0)=y_0'$

point

slope

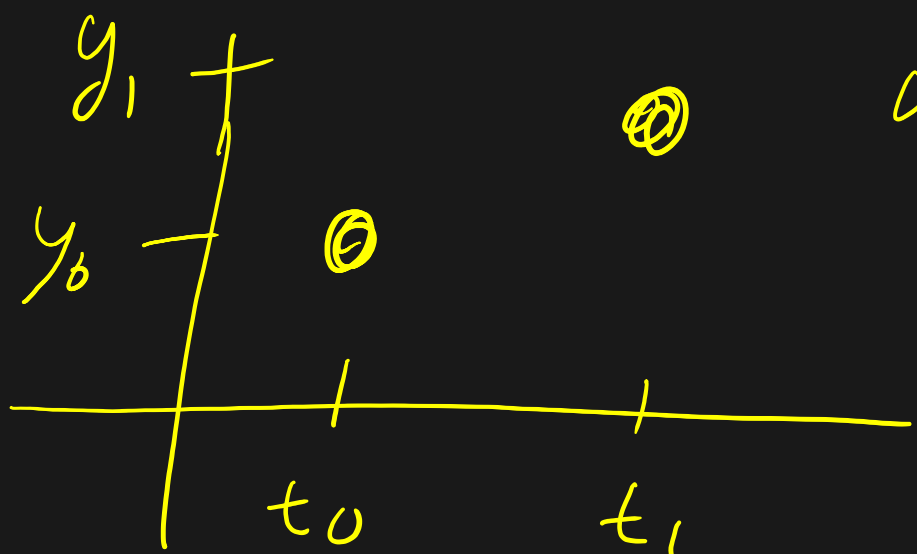



slope at $(t_0, y_0)$ is $y_0' = y'(t_0)$

## 2nd order LINEAR differential equation:

Thm 3.2.1: If $p:(a,b)\to R$, $q:(a,b)\to R$, and $g:(a,b)\to R$ are continuous and $a<t_0<b$, then $\exists!$ function $y=\phi(t)$, $\phi:(a,b)\to R$ that satisfies

$$\text{IVP: } y''+p(t)y'+q(t)y=g(t),\ \ y(t_0)=y_0,\ \ \underline{y'(t_0)=y'_0}$$




when you solve IVP, not there may not be a soln passing thru 2 given pts

IVP is <u>NOT</u> $(t_0, y_0)$ $(t_1, y_1)$

Thm 3.2.2: If $\phi_1$ and $\phi_2$ are two solutions to a homogeneous linear differential equation

$$y'' + p(t)y' + q(t)y = 0$$

then $\boxed{c_1\phi_1 + c_2\phi_2}$ is also a solution to this linear differential equation.

Linear comb of soln to homog LINEAR DE are also sol'n

Proof of thm 3.2.2:

Since $y(t) = \phi_i(t)$ is a solution to the linear homogeneous differential equation $y'' + py' + qy = 0$ where $p$ and $q$ are functions of $t$ (note this includes the case with constant coefficients), then

hypothesis:

$$\begin{cases} \phi_1'' + p(t)\phi_1' + q(t)\phi_1 = 0 \\ \phi_2'' + p(t)\phi_2' + q(t)\phi_2 = 0 \end{cases}$$

Claim: $y(t) = c_1\phi_1(t) + c_2\phi_2(t)$ is also a solution to $y'' + py' + qy = 0$

Pf of claim: Plug into LHS

$$(c_1\phi_1 + c_2\phi_2)'' + p(c_1\phi_1 + c_2\phi_2)' + q(c_1\phi_1 + c_2\phi_2)$$

$$= c_1\phi_1'' + c_2\phi_2'' + p(c_1\phi_1' + c_2\phi_2') + q(c_1\phi_1 + c_2\phi_2)$$

$$= c_1\phi_1'' + c_2\phi_2'' + pc_1\phi_1' + pc_2\phi_2' + qc_1\phi_1 + qc_2\phi_2$$

$$= c_1(\phi_1'' + p\phi_1' + q\phi_1) + c_2(\phi_2'' + p\phi_2' + q\phi_2)$$

$$= c_1(0) + c_2(0) = 0 \quad \blacksquare$$

RHS

Claim: If $y = \phi_1(t)$ and $y = \phi_2(t)$ are linearly independent solutions to $y'' + py' + qy = 0$ , then

general solution is $\boxed{y(t) = c_1\phi_1(t) + c_2\phi_2(t)}$

That is any solution to this linear 2nd order homogeneous differential equation can be written as a linear combination of the linear independent functions $y = \phi_1(t)$ and $y = \phi_2(t)$.

If we know soln is unique $\Leftarrow$ claim

Thm 3.2.2 $\Rightarrow$ If $\phi_1$ & $\phi_2$ solns $\Rightarrow$ $c_1\phi_1 + c_2\phi_2$ is a soln

Claim $\Rightarrow$ If $y = f(t)$ is a sol'n then $f(t) = c_1\phi_1(t) + c_2\phi_2(t)$

Claim: If $y = \phi_1(t)$ and $y = \phi_2(t)$ are linearly independent solutions to $y'' + py' + qy = 0$ , then

general solution is $y(t) = c_1\phi_1(t) + c_2\phi_2(t)$

That is any solution to this linear 2nd order homogeneous differential equation can be written as a linear combination of the linear independent functions $y = \phi_1(t)$ and $y = \phi_2(t)$.

To solve $2^{nd}$ $\boxed{(n^{th}}$ order linear homog DE

We just need to find

2 $\boxed{(n)}$ linearly independent soln's

ch 4 $\rightarrow$

Derivation of general solutions:

Solve $ay'' + by' + cy = 0$. Educated guess $y = e^{rt}$, then

$ar^2e^{rt} + bre^{rt} + ce^{rt} = 0$ implies $ar^2 + br + c = 0$,

Suppose $r = r_1, r_2$ are solutions to $ar^2 + br + c = 0$

$$r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If $\boxed{b^2 - 4ac > 0}$ we guessed $e^{rt}$ is a solution and noted that any linear combination of solutions is a solution to a homogeneous linear differential equation.

2 real solns

claim $\Rightarrow$ general soln is $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

need 2 l.i. solns

Claim: $y = e^{dt}\cos(nt)$ and $y = e^{dt}\sin(nt)$ are 2 l.i. soln

Section 3.3: If $b^2 - 4ac < 0$, :

Changed format of $y = c_1e^{r_1t} + c_2e^{r_2t}$ to linear combination of real-valued functions instead of complex valued functions by using Euler's formula:

$$e^{it} = cos(t) + isin(t)$$

Hence $e^{(d+in)t} = e^{dt}e^{int} = e^{dt}[cos(nt) + isin(nt)]$

Let $r_1 = d + in$, $r_2 = d - in$

$r = d \pm in$

complex conjugate

non simplified general sol'n

Not acceptable soln

It is correct but not simplified

$$y = c_1 e^{(d+in)t} + c_2 e^{(d-in)t} = c_1 e^{dt+int} + c_2 e^{dt-int}$$

$e^{i(-nt)}$

$$= c_1 e^{dt} e^{int} + c_2 e^{dt} e^{-int}$$

euler's formula

$$= c_1 e^{dt}[cos(nt) + isin(nt)] + c_2 e^{dt}[cos(-nt) + isin(-nt)]$$

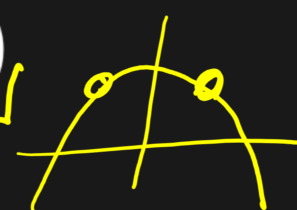
$$= c_1 e^{dt}cos(nt) + ic_1 e^{dt}sin(nt) + c_2 e^{dt}cos(nt) - ic_2 e^{dt}sin(nt)$$

$$= (c_1 + c_2)e^{dt}cos(nt) + i(c_1 - c_2)e^{dt}sin(nt)$$

$$= k_1 e^{dt}cos(nt) + k_2 e^{dt}sin(nt)$$

use this

simplied version of general soln



cosine even fn

sin is an odd fn

$e^{-r_2 t}$ $e^{r_1 t}$

$$r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = d \pm int$$

Alternate proof using linearity:

did this last end of class Friday

$y = e^{dt+int} = e^{dt}[cos(nt) + isin(nt)]$ and

$y = e^{dt-int} = e^{dt}[cos(-nt) + isin(-nt)] = e^{dt}[cos(nt) - isin(nt)]$

are solutions

Linear combinations of solutions are solutions:

$y = e^{dt}[cos(nt) + \cancel{isin(nt)}] + e^{dt}[cos(nt) - \cancel{isin(nt)}]$

$y = e^{dt}[\cancel{cos(nt)} + isin(nt)] - e^{dt}[\cancel{cos(nt)} - isin(nt)]$

Thus $\boxed{y = 2e^{dt}cos(nt)}$ and $y = \boxed{2ie^{dt}sin(nt)}$ are both solutions

addition subtration

3.3: complex ~~roots~~ 2nd order | Looking for 2 linearly indep soln

Since $y = 2e^{dt}cos(nt)$ and $y = 2ie^{dt}sin(nt)$ are solutions to $ay'' + by' + cy = 0$ where $b^2 - 4ac < 0$,

$$\Rightarrow y = \frac{1}{2}\left(2e^{dt}\cos(nt)\right) = e^{dt}\cos(nt)$$

and $$y = \frac{1}{2i}\left(2ie^{dt}\sin(nt)\right) = e^{dt}\sin(nt)$$

$$\Rightarrow \boxed{y = c_1 e^{dt}\cos(nt) + c_2 e^{dt}\sin(nt)}$$

is a general soln

since $y = e^{dt}\cos(nt)$ & $y = e^{dt}\sin(nt)$ are linearly indep fcn and are soln

$r = r_1 = r_2$ 1 repeated root

Section 3.4: If $b^2 - 4ac = 0$, then $r_1 = r_2$.
Hence one sol'n is $y = e^{r_1 t}$ Need $2^{nd}$ sol'n to $\boxed{ay'' + by' + cy = 0.}$

If $y = e^{rt}$ is a solution, $y = ce^{rt}$ is a solution.

How about $\boxed{y = v(t)e^{rt}?}$ check by plugging in

$$y' = v'(t)e^{rt} + v(t)re^{rt}$$

$$y'' = v''(t)e^{rt} + v'(t)re^{rt} + v'(t)re^{rt} + v(t)r^2e^{rt}$$

$$= v''(t)e^{rt} + 2v'(t)re^{rt} + v(t)r^2e^{rt}$$

$$a(v''e^{rt} + 2v're^{rt} + vr^2e^{rt}) + b(v'e^{rt} + vre^{rt}) + cve^{rt} = 0$$

$$a(v''(t) + 2v'(t)r + v(t)r^2) + b(v'(t) + v(t)r) + cv(t) = 0$$

$$av''(t) + 2av'(t)r + av(t)r^2 + bv'(t) + bv(t)r + cv(t) = 0$$

$$av''(t) + (2ar + b)v'(t) + \cancel{(ar^2 + br + c)}^{0}v(t) = 0$$

$ar^2 + br + c = 0$

$$av''(t) + (2ar + b)v'(t) + (ar^2 + br + c)v(t) = 0$$

$$av''(t) + (\cancel{2}a(\frac{-b}{\cancel{2}a}) + b)v'(t) + 0 = 0$$

$r = \frac{-b \pm \sqrt{0}}{2a}$

since $ar^2 + br + c = 0$ and $\boxed{r = \frac{-b}{2a}}$

$av''(t) + \cancel{(-b + b)v'(t)} = 0.$ Thus $av''(t) = 0.$

Hence $v''(t) = 0$ and $v'(t) = k_1$ and $v(t) = k_1 t + k_2$

calculus 1

Hence $v(t)e^{r_1 t} = (k_1 t + k_2)e^{r_1 t}$ is a soln

Let $k_1 = 1$

$k_2 = 0$

Thus $te^{r_1 t}$ is a nice second solution.

Hence general solution is $\boxed{y = c_1 e^{r_1 t} + c_2 t e^{r_1 t}}$

## Section 3.4: Reduction of order

Suppose $y = \phi_1(t)$ is a solution to $y'' + p(t)y' + q(t)y = 0$

Guess $y = v(t)\phi_1(t)$ is also a solution.

Solve for unknown function $v(t)$ by plugging in:

$y = \phi_1(t)$ is a solution to $y'' + p(t)y' + q(t)y = 0$
implies $\phi_1'' + p(t)\phi_1' + q(t)\phi_1 = 0$

$$y = v(t)\phi_1(t) \implies y' = v'(t)\phi_1(t) + v(t)\phi_1'(t)$$

$$\begin{aligned} y'' &= v''(t)\phi_1(t) + v'(t)\phi_1'(t) + v'(t)\phi_1'(t) + v(t)\phi_1''(t) \\ &= v''(t)\phi_1(t) + 2v'(t)\phi_1'(t) + v(t)\phi_1''(t) \end{aligned}$$

$$y'' + p(t)y' + q(t)y = 0$$

---

$$v''(t)\phi_1(t) \;+\; 2v'(t)\phi_1'(t) \;+\; v(t)\phi_1''(t)$$

$$+\; p(t)[v'(t)\phi_1(t) + v(t)\phi_1'(t)]$$

$$+\; q(t)[v(t)\phi_1(t)] = 0$$

## Section 3.4: Reduction of order

Suppose $y = \phi_1(t)$ is a solution to $y'' + p(t)y' + q(t)y = 0$

Guess $y = v(t)\phi_1(t)$ is also a solution.

Solve for unknown function $v(t)$ by plugging in:

---

$$v''(t)\phi_1(t) \ + \ 2v'(t)\phi_1'(t) \ + \ p(t)v'(t)\phi_1(t) = 0$$

# Section 3.4: Reduction of order

Suppose $y = \phi_1(t)$ is a solution to $y'' + p(t)y' + q(t)y = 0$

Guess $y = v(t)\phi_1(t)$ is also a solution.

Solve for unknown function $v(t)$ by plugging in:

---

$$v''(t)\phi_1(t) \ + \ v'(t)[2\phi_1'(t) \ + \ p(t)\phi_1(t)] = 0$$

$$v''(t)\phi_1(t) \; + \; v'(t)[2\phi_1'(t) \; + \; p(t)\phi_1(t)] = 0$$

3.1 # 21 $ay'' + by' + cy = 0$, $a>0$

$ar^2 + br + c = 0$

$r = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

will use in ch 7

case 2 $b<0$.

$-b+\sqrt{b^2-4ac} > 0$

Need $\underbrace{-b}_{\text{positive}} - \underbrace{\sqrt{b^2-4ac}}_{\text{positive}} < 0$

b) Want 2 real sol'n where one negative ✓ & one is positive

real $\Rightarrow$ $b^2-4ac > 0$

since $a>0$ $\Rightarrow$ $\dfrac{-b-\sqrt{b^2-4ac}}{2a} = \dfrac{-}{+} < 0$

case 1 $b>0$

want $\underbrace{-b+\sqrt{b^2-4ac}}$ $> 0$ $\Rightarrow$ $c<0$

$0 \le \sqrt{b^2-4ac}$ want $> \sqrt{b^2}$

$\sqrt{b^2-4ac} > b$

If $c<0$

$\sqrt{b^2-4ac} > \sqrt{b^2}$