Summary of sections 3.1, 3, 4:
Solve linear homogeneous 2nd order DE with constant coefficients.

Solve $ay'' + by' + cy = 0$. Educated guess $y = e^{rt}$, then

$ar^2e^{rt} + bre^{rt} + ce^{rt} = 0$ implies $ar^2 + br + c = 0$,

Suppose $r = r_1, r_2$ are solutions to $ar^2 + br + c = 0$

$$r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If $r_1 \neq r_2$, then $b^2 - 4ac \neq 0$.

Hence a general solution is $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

If $b^2 - 4ac > 0$, general sol'n is $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$.

If $b^2 - 4ac < 0$, change format to linear combination of real-valued functions instead of complex valued functions by using Euler's formula.

general solution is $y = c_1 e^{dt} cos(nt) + c_2 e^{dt} sin(nt)$ where $r = d \pm in$

If $b^2 - 4ac = 0$, $r_1 = r_2$, so need 2nd (independent) solution: $te^{r_1 t}$

Hence general solution is $y = c_1 e^{r_1 t} + c_2 t e^{r_1 t}$.

Initial value problem: use $y(t_0) = y_0$, $y'(t_0) = y'_0$ to solve for c_1, c_2 to find unique solution.

1st order LINEAR differential equation:

Thm 2.4.1: If $p : (a, b) \to R$ and $g : (a, b) \to R$ are continuous and $a < t_0 < b$, then there exists a unique function $y = \phi(t), \ \ \phi : (a, b) \to R$ that satisfies the

$$\text{IVP: } y' + p(t)y = g(t), \quad y(t_0) = y_0$$

Proof 1: Constructive proof (use integrating factor to find solution).

Proof 2 outline: Use linearity.

1st order LINEAR differential equation:

Thm 2.4.1: If $p : (a, b) \to R$ and $g : (a, b) \to R$ are continuous and $a < t_0 < b$, then there exists a unique function $y = \phi(t), \ \ \phi : (a, b) \to R$ that satisfies the

$$\text{IVP: } y' + p(t)y = g(t), \quad y(t_0) = y_0$$

Thm: If $y = \phi_1(t)$ is a solution to homogeneous equation, $y' + p(t)y = 0$, then $y = c\phi_1(t)$ is the general solution to this equation.

If in addition $y = \psi(t)$ is a solution to non-homogeneous equation, $y' + p(t)y = g(t)$, then $y = c\phi_1(t) + \psi(t)$ is the general solution to this equation.

1st order LINEAR differential equation:

Thm 2.4.1: If $p : (a, b) \to R$ and $g : (a, b) \to R$ are continuous and $a < t_0 < b$, then
$\exists!$ function $y = \phi(t)$, $\phi : (a, b) \to R$ that satisfies

$$\text{IVP: } y' + p(t)y = g(t), \quad y(t_0) = y_0$$

2nd order LINEAR differential equation:

Thm 3.2.1: If $p : (a, b) \to R$, $q : (a, b) \to R$, and $g : (a, b) \to R$ are continuous and $a < t_0 < b$, then
$\exists!$ function $y = \phi(t)$, $\phi : (a, b) \to R$ that satisfies

$$\text{IVP: } y'' + p(t)y' + q(t)y = g(t), \;\; y(t_0) = y_0, \;\; y'(t_0) = y'_0$$

2nd order LINEAR differential equation:

Thm 3.2.1: If $p : (a, b) \to R$, $q : (a, b) \to R$, and $g : (a, b) \to R$ are continuous and $a < t_0 < b$, then $\exists!$ function $y = \phi(t)$, $\phi : (a, b) \to R$ that satisfies

IVP: $y'' + p(t)y' + q(t)y = g(t), \;\; y(t_0) = y_0, \;\; y'(t_0) = y'_0$

2nd order LINEAR differential equation:

Thm 3.2.1: If $p : (a, b) \to R$, $q : (a, b) \to R$, and $g : (a, b) \to R$ are continuous and $a < t_0 < b$, then $\exists!$ function $y = \phi(t)$, $\phi : (a, b) \to R$ that satisfies

IVP: $y'' + p(t)y' + q(t)y = g(t), \;\; y(t_0) = y_0, \;\; y'(t_0) = y'_0$

Thm 3.2.2: If ϕ_1 and ϕ_2 are two solutions to a <u>homogeneous</u> linear differential equation

$$y'' + p(t)y' + q(t)y = 0$$

then $c_1\phi_1 + c_2\phi_2$ is also a solution to this linear differential equation.

Proof of thm 3.2.2:

Since $y(t) = \phi_i(t)$ is a solution to the linear homogeneous differential equation $y'' + py' + qy = 0$ where p and q are functions of t (note this includes the case with constant coefficients), then

Claim: $y(t) = c_1\phi_1(t) + c_2\phi_2(t)$ is also a solution to $y'' + py' + qy = 0$

Pf of claim:

Claim: If $y = \phi_1(t)$ and $y = \phi_2(t)$ are linearly independent solutions to $y'' + py' + qy = 0$, then

$$\text{general solution is } y(t) = c_1\phi_1(t) + c_2\phi_2(t)$$

That is any solution to this linear 2nd order homogeneous differential equation can be written as a linear combination of the linear independent functions $y = \phi_1(t)$ and $y = \phi_2(t)$.

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That is any solution to this linear 2nd order homogeneous differential equation can be written as a linear combination of the linear independent functions $y = \phi_1(t)$ and $y = \phi_2(t)$.

Derivation of general solutions:

Solve $ay'' + by' + cy = 0$. Educated guess $y = e^{rt}$, then

$ar^2e^{rt} + bre^{rt} + ce^{rt} = 0$ implies $ar^2 + br + c = 0$,

Suppose $r = r_1, r_2$ are solutions to $ar^2 + br + c = 0$

$$r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If $b^2 - 4ac > 0$ we guessed e^{rt} is a solution and noted that any linear combination of solutions is a solution to a homogeneous linear differential equation.

Section 3.3: If $b^2 - 4ac < 0$, :

Changed format of $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ to linear combination of real-valued functions instead of complex valued functions by using Euler's formula:

$$e^{it} = cos(t) + isin(t)$$

Hence $e^{(d+in)t} = e^{dt}e^{int} = e^{dt}[cos(nt) + isin(nt)]$

Let $r_1 = d + in$, $r_2 = d - in$

$$y = c_1 e^{(d+in)t} + c_2 e^{(d-in)t} \;=\; c_1 e^{dt+int} + c_2 e^{dt-int}$$

$$= c_1 e^{dt} e^{int} + c_2 e^{dt} e^{-int}$$

$$= c_1 e^{dt}[cos(nt) + isin(nt)] + c_2 e^{dt}[cos(-nt) + isin(-nt)]$$

$$= c_1 e^{dt} cos(nt) + ic_1 e^{dt} sin(nt) + c_2 e^{dt} cos(nt) - ic_2 e^{dt} sin(nt)$$

$$=(c_1 + c_2) e^{dt} cos(nt) + i(c_1 - c_2) e^{dt} sin(nt)$$

$$= k_1 e^{dt} cos(nt) + k_2 e^{dt} sin(nt)$$

$$r_1, r_2 = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = d \pm int$$

Alternate proof using linearity:

$y = e^{dt+int} = e^{dt}[cos(nt) + isin(nt)]$ and

$y = e^{dt-int} = e^{dt}[cos(-nt) + isin(-nt)] = e^{dt}[cos(nt) - isin(nt)]$
are solutions

Linear combinations of solutions are solutions:

$$y = e^{dt}[cos(nt) + isin(nt)] \; + \; e^{dt}[cos(nt) - isin(nt)]$$

$$y = e^{dt}[cos(nt) + isin(nt)] \; - \; e^{dt}[cos(nt) - isin(nt)]$$

Thus $y = 2e^{dt}cos(nt)$ and $y = 2ie^{dt}sin(nt)$ are both solutions

Since $y = 2e^{dt}cos(nt)$ and $y = 2ie^{dt}sin(nt)$ are solutions to $ay'' + by' + cy = 0$ where $b^2 - 4ac < 0$,

Section 3.4: If $b^2 - 4ac = 0$, then $r_1 = r_2$.
Hence one sol'n is $y = e^{r_1 t}$ Need 2^{nd} sol'n to $ay'' + by' + cy = 0$.

If $y = e^{rt}$ is a solution, $y = ce^{rt}$ is a solution.

How about $y = v(t)e^{rt}$?

$$y' = v'(t)e^{rt} + v(t)re^{rt}$$

$$y'' = v''(t)e^{rt} + v'(t)re^{rt} + v'(t)re^{rt} + v(t)r^2e^{rt}$$
$$= v''(t)e^{rt} + 2v'(t)re^{rt} + v(t)r^2e^{rt}$$

$$a(v''e^{rt} + 2v're^{rt} + vr^2e^{rt}) + b(v'e^{rt} + vre^{rt}) + cve^{rt} = 0$$

$$a(v''(t) + 2v'(t)r + v(t)r^2) + b(v'(t) + v(t)r) + cv(t) = 0$$

$$av''(t) + 2av'(t)r + av(t)r^2 + bv'(t) + bv(t)r + cv(t) = 0$$

$$av''(t) + (2ar + b)v'(t) + (ar^2 + br + c)v(t) = 0$$

$$av''(t) + (2ar + b)v'(t) + (ar^2 + br + c)v(t) = 0$$

$$av''(t) + (2a(\tfrac{-b}{2a}) + b)v'(t) + 0 = 0$$

since $ar^2 + br + c = 0$ and $r = \frac{-b}{2a}$

$av''(t) + (-b + b)v'(t) = 0.$ Thus $av''(t) = 0.$

Hence $v''(t) = 0$ and $v'(t) = k_1$ and $v(t) = k_1 t + k_2$

Hence $v(t)e^{r_1 t} = (k_1 t + k_2)e^{r_1 t}$ is a soln

Thus $te^{r_1 t}$ is a nice second solution.

Hence general solution is $y = c_1 e^{r_1 t} + c_2 t e^{r_1 t}$

Section 3.4: Reduction of order

Suppose $y = \phi_1(t)$ is a solution to $y'' + p(t)y' + q(t)y = 0$

Guess $y = v(t)\phi_1(t)$ is also a solution.

Solve for unknown function $v(t)$ by plugging in:

$y = \phi_1(t)$ is a solution to $y'' + p(t)y' + q(t)y = 0$
implies $\phi_1'' + p(t)\phi_1' + q(t)\phi_1 = 0$

$$y = v(t)\phi_1(t) \implies y' = v'(t)\phi_1(t) + v(t)\phi_1'(t)$$

$$y'' = v''(t)\phi_1(t) + v'(t)\phi_1'(t) + v'(t)\phi_1'(t) + v(t)\phi_1''(t)$$

$$= v''(t)\phi_1(t) + 2v'(t)\phi_1'(t) + v(t)\phi_1''(t)$$

$$y'' + p(t)y' + q(t)y = 0$$

$$v''(t)\phi_1(t) \; + \; 2v'(t)\phi_1'(t) \; + \; v(t)\phi_1''(t)$$

$$+ \; p(t)[v'(t)\phi_1(t) + v(t)\phi_1'(t)]$$

$$+ \; q(t)[v(t)\phi_1(t)] = 0$$

Section 3.4: Reduction of order

Suppose $y = \phi_1(t)$ is a solution to $y'' + p(t)y' + q(t)y = 0$

Guess $y = v(t)\phi_1(t)$ is also a solution.

Solve for unknown function $v(t)$ by plugging in:

$$v''(t)\phi_1(t) \; + \; 2v'(t)\phi_1'(t) \; + \; p(t)v'(t)\phi_1(t) = 0$$

Section 3.4: Reduction of order

Suppose $y = \phi_1(t)$ is a solution to $y'' + p(t)y' + q(t)y = 0$

Guess $y = v(t)\phi_1(t)$ is also a solution.

Solve for unknown function $v(t)$ by plugging in:

$$v''(t)\phi_1(t) \ + \ v'(t)[2\phi_1'(t) \ + \ p(t)\phi_1(t)] = 0$$

$$v''(t)\phi_1(t) \; + \; v'(t)[2\phi_1'(t) \; + \; p(t)\phi_1(t)] = 0$$

Solve: $y'' + y = 0,\ y(0) = -1,\ y'(0) = -3$

Solve: $y'' + y = 0$, $y(0) = -1$, $y'(0) = -3$

$r^2 + 1 = 0$ implies $r^2 = -1$. Thus $r = \pm i$.

NOT RECOMMENDED: work with $y = c_1 e^{it} + c_2 e^{-it}$

$$y' = ic_1 e^{it} - ic_2 e^{-it}$$

$y(0) = -1$: $\quad -1 = c_1 e^0 + c_2 e^0$ implies $-1 = c_1 + c_2$.

$y'(0) = -3$: $\quad -3 = ic_1 e^0 - ic_2 e^0$ implies $-3 = ic_1 - ic_2$.

$$-1i = ic_1 + ic_2.$$

$$-3 = ic_1 - ic_2.$$

$-1i = ic_1 + ic_2.$

$-3 = ic_1 - ic_2.$

$2ic_1 = -3 - i$ implies $c_1 = \frac{-3i - i^2}{-2} = \frac{3i-1}{2}$

$2ic_2 = 3 - i$ implies $c_2 = \frac{3i - i^2}{-2} = \frac{-3i-1}{2}$

Euler's formula: $e^{ix} = cos(x) + isin(x)$

$$y = (\frac{3i-1}{2})e^{it} + (\frac{-3i-1}{2})e^{-it}$$

$$= (\frac{3i-1}{2})[cos(t) + isin(t)] + (\frac{-3i-1}{2})[cos(-t) + isin(-t)]$$

$$= (\frac{3i-1}{2})[cos(t) + isin(t)] + (\frac{-3i-1}{2})[cos(t) - isin(t)]$$

$$= (\frac{3i}{2})cos(t) + (\frac{3i}{2})isin(t) + (\frac{-1}{2})cos(t) + (\frac{-1}{2})isin(t)$$

$$+(\frac{-3i}{2})cos(t) - (\frac{-3i}{2})isin(t) + (\frac{-1}{2})cos(t) - (\frac{-1}{2})isin(t)$$

$$= (\frac{3i}{2})isin(t) + (\frac{-1}{2})cos(t) + (\frac{3i}{2})isin(t) + (\frac{-1}{2})cos(t)$$

$$= -(\frac{3}{2})sin(t) - (\frac{1}{2})cos(t) - (\frac{3}{2})sin(t) - (\frac{1}{2})cos(t)$$

$$= -3sin(t) - 1cos(t)$$

Solve: $y'' + y = 0$, $y(0) = -1$, $y'(0) = -3$

$r^2 + 1 = 0$ implies $r^2 = -1$. Thus $r = \pm i$.

RECOMMENDED Method:

Since $r = 0 \pm 1i$, $y = c_1 cos(t) + c_2 sin(t)$

Then $y' = -c_1 sin(t) + c_2 cos(t)$

$y(0) = -1$: $\quad -1 = c_1 cos(0) + c_2 sin(0)$ implies $-1 = c_1$

$y'(0) = -3$: $\quad -3 = -c_1 sin(0) + c_2 cos(0)$ implies $-3 = c_2$

Thus IVP solution: $y = -cos(t) - 3sin(t)$

When does the following IVP have unique sol'n:

IVP: $ay'' + by' + cy = 0$, $y(t_0) = y_0$, $y'(t_0) = y_1$.

Suppose $y = c_1\phi_1(t) + c_2\phi_2(t)$ is a solution to $ay'' + by' + cy = 0$.

Then $y' = c_1\phi_1'(t) + c_2\phi_2'(t)$

$y(t_0) = y_0$: $y_0 = c_1\phi_1(t_0) + c_2\phi_2(t_0)$

$y'(t_0) = y_1$: $y_1 = c_1\phi_1'(t_0) + c_2\phi_2'(t_0)$

To find IVP solution, need to solve above system of two equations for the unknowns c_1 and c_2.

Note the IVP has a unique solution if and only if the above system of two equations has a unique solution for c_1 and c_2.

$$\begin{aligned} y(t_0) = y_0: \qquad & y_0 = c_1\phi_1(t_0) + c_2\phi_2(t_0) \\ y'(t_0) = y_1: \qquad & y_1 = c_1\phi_1'(t_0) + c_2\phi_2'(t_0) \end{aligned}$$

Note that in these equations c_1 and c_2 are the unknowns and $y_0, \phi_1(t_0), \phi_2(t_0),\ y_1, \phi_1'(t_0), \phi_2'(t_0)$ are the constants. We can translate this linear system of equations into matrix form:

Note this equation has a unique solution if and only if

Definition: The Wronskian of two differential functions, ϕ_1 and ϕ_2 is

$$W(\phi_1, \phi_2) = \phi_1 \phi_2' - \phi_1' \phi_2 =$$

Definition: The Wronskian of two differential functions, ϕ_1 and ϕ_2 is

$$W(\phi_1, \phi_2) = \phi_1\phi_2' - \phi_1'\phi_2 = \begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix}$$

Examples:

1.) $W(cos(t), sin(t)) = \begin{vmatrix} cos(t) & sin(t) \\ -sin(t) & cos(t) \end{vmatrix}$

$$= cos^2(t) + sin^2(t) = 1 > 0.$$

2.) W$(e^{dt}cos(nt), e^{dt}sin(nt))$

$$= \begin{vmatrix} e^{dt}cos(nt) & e^{dt}sin(nt) \\ de^{dt}cos(nt) - ne^{dt}sin(nt) & de^{dt}sin(nt) + ne^{dt}cos(nt) \end{vmatrix}$$

$$=e^{dt}cos(nt)(de^{dt}sin(nt)+ne^{dt}cos(nt))-e^{dt}sin(nt)(de^{dt}cos(nt)-ne^{dt}sin(nt))$$

$$=e^{2dt}[cos(nt)(dsin(nt)+ncos(nt))-sin(nt)(dcos(nt)-nsin(nt))]$$

$$=e^{2dt}[dcos(nt)sin(nt)+ncos^2(nt)-dsin(nt)cos(nt)+nsin^2(nt)])$$

$$= e^{2dt}[ncos^2(nt) + nsin^2(nt)] = ne^{2dt}[cos^2(nt) + sin^2(nt)]$$

$= ne^{2dt} > 0$ for all t.