Summary of sections 3.1, 3, 4:
Solve linear homogeneous 2nd order DE with constant coefficients.

Solve $ay'' + by' + cy = 0$. Educated guess $y = e^{rt}$, then

$ar^2e^{rt} + bre^{rt} + ce^{rt} = 0$ implies $ar^2 + br + c = 0$,

Suppose $r = r_1, r_2$ are solutions to $ar^2 + br + c = 0$

$$r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If $r_1 \neq r_2$, then $b^2 - 4ac \neq 0$.

Hence a general solution is $y = c_1e^{r_1t} + c_2e^{r_2t}$

If $b^2 - 4ac > 0$, general sol'n is $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$.

If $b^2 - 4ac < 0$, change format to linear combination of real-valued functions instead of complex valued functions by using Euler's formula.

general solution is $y = c_1 e^{dt} cos(nt) + c_2 e^{dt} sin(nt)$ where $r = d \pm in$

If $b^2 - 4ac = 0$, $r_1 = r_2$, so need 2nd (independent) solution: $te^{r_1 t}$

Hence general solution is $y = c_1 e^{r_1 t} + c_2 t e^{r_1 t}$.

Initial value problem: use $y(t_0) = y_0$, $y'(t_0) = y'_0$ to solve for c_1, c_2 to find unique solution.

Ex 1: Solve $y'' - 3y' - 4y = 0$, $y(0) = 1$, $y'(0) = 2$.

Ex 1: Solve $y'' - 3y' - 4y = 0, \;\; y(0) = 1, \, y'(0) = 2.$

If $y = e^{rt}$, then $y' = re^{rt}$ and $y'' = r^2e^{rt}$.

$r^2 - 3r - 4 = 0 \Rightarrow (r-4)(r+1) = 0 \Rightarrow r = 4, -1.$

Hence general solution is $y = c_1e^{4t} + c_2e^{-t}$

Ex 1: Solve $y'' - 3y' - 4y = 0$, $y(0) = 1$, $y'(0) = 2$.

If $y = e^{rt}$, then $y' = re^{rt}$ and $y'' = r^2e^{rt}$.

$$r^2 - 3r - 4 = 0 \Rightarrow (r-4)(r+1) = 0 \Rightarrow r = 4, -1.$$

Hence general solution is $y = c_1e^{4t} + c_2e^{-t}$

Solution to IVP: Need to solve for 2 unknowns, c_1 & c_2

Ex 1: Solve $y'' - 3y' - 4y = 0$, $y(0) = 1$, $y'(0) = 2$.

If $y = e^{rt}$, then $y' = re^{rt}$ and $y'' = r^2e^{rt}$.

$$r^2 - 3r - 4 = 0 \Rightarrow (r-4)(r+1) = 0 \Rightarrow r = 4, -1.$$

Hence general solution is $y = c_1e^{4t} + c_2e^{-t}$

Solution to IVP: Need to solve for 2 unknowns, c_1 & c_2

Thus need 2 eqns:

$y = c_1e^{4t} + c_2e^{-t}$, $\quad y(0) = 1 \quad \Rightarrow \quad 1 = c_1 + c_2$

$y' = 4c_1e^{4t} - c_2e^{-t}$, $\quad y'(0) = 2 \quad \Rightarrow \quad 2 = 4c_1 - c_2$

$3 = 5c_1 \Rightarrow c_1 = \frac{3}{5}$ and $c_2 = 1 - c_1 = 1 - \frac{3}{5} = \frac{2}{5}$

Thus IVP soln: $y = \frac{3}{5}e^{4t} + \frac{2}{5}e^{-t}$

Ex 2: Solve $y'' - 3y' + 4y = 0.$

$y = e^{rt}$ implies $r^2 - 3r + 4 = 0$ and hence

Ex 2: Solve $y'' - 3y' + 4y = 0$.

$y = e^{rt}$ implies $r^2 - 3r + 4 = 0$ and hence

$$r = \frac{3 \pm \sqrt{(-3)^2 - 4(1)(4)}}{2} = \frac{3}{2} \pm \frac{\sqrt{9-16}}{2} = \frac{3}{2} \pm i\frac{\sqrt{7}}{2}$$

Hence general sol'n is $y = c_1 e^{\frac{3}{2}t} cos(\frac{\sqrt{7}}{2}t) + c_2 e^{\frac{3}{2}t} sin(\frac{\sqrt{7}}{2}t)$

Ex 3: $y'' - 6y' + 9y = 0$

Ex 3: $y'' - 6y' + 9y = 0$

$$r^2 - 6r + 9 = (r-3)^2 = 0$$

Repeated root, $r = 3$ implies

general solution is $y = c_1 e^{3t} + c_2 t e^{3t}$

Homogeneous linear 2nd order differential equation

$$R(t)y'' + P(t)y' + Q(t)y = 0$$

Existence and Uniqueness for LINEAR DEs.

Homogeneous:

$$y^{(n)} + p_1(t)y^{(n-1)} + ...p_{n-1}(t)y' + p_n(t)y = 0$$

Non-homogeneous: $g(t) \neq 0$

$$y^{(n)} + p_1(t)y^{(n-1)} + ...p_{n-1}(t)y' + p_n(t)y = g(t)$$

1st order LINEAR differential equation:

Thm 2.4.1: If $p : (a, b) \to R$ and $g : (a, b) \to R$ are continuous and $a < t_0 < b$, then there exists a unique function $y = \phi(t),\ \ \phi : (a, b) \to R$ that satisfies the

$$\text{IVP: } y' + p(t)y = g(t), \quad y(t_0) = y_0$$

Proof 1: Constructive proof (use integrating factor to find solution).

Proof 2 outline: Use linearity.

1st order LINEAR differential equation:

Thm 2.4.1: If $p : (a, b) \to R$ and $g : (a, b) \to R$ are continuous and $a < t_0 < b$, then there exists a unique function $y = \phi(t),\ \ \phi : (a, b) \to R$ that satisfies the

$$\text{IVP: } y' + p(t)y = g(t), \quad y(t_0) = y_0$$

Thm: If $y = \phi_1(t)$ is a solution to homogeneous equation, $y' + p(t)y = 0$, then $y = c\phi_1(t)$ is the general solution to this equation.

If in addition $y = \psi(t)$ is a solution to non-homogeneous equation, $y' + p(t)y = g(t)$, then $y = c\phi_1(t) + \psi(t)$ is the general solution to this equation.

1st order LINEAR differential equation:

Thm 2.4.1: If $p : (a, b) \to R$ and $g : (a, b) \to R$ are continuous and $a < t_0 < b$, then $\exists!$ function $y = \phi(t)$, $\phi : (a, b) \to R$ that satisfies

$$\text{IVP: } y' + p(t)y = g(t), \quad y(t_0) = y_0$$

2nd order LINEAR differential equation:

Thm 3.2.1: If $p : (a, b) \to R$, $q : (a, b) \to R$, and $g : (a, b) \to R$ are continuous and $a < t_0 < b$, then $\exists!$ function $y = \phi(t)$, $\phi : (a, b) \to R$ that satisfies

$$\text{IVP: } y'' + p(t)y' + q(t)y = g(t), \;\; y(t_0) = y_0, \;\; y'(t_0) = y'_0$$

2nd order LINEAR differential equation:

Thm 3.2.1: If $p : (a, b) \to R$, $q : (a, b) \to R$, and $g : (a, b) \to R$ are continuous and $a < t_0 < b$, then $\exists!$ function $y = \phi(t)$, $\phi : (a, b) \to R$ that satisfies

IVP: $y'' + p(t)y' + q(t)y = g(t), \;\; y(t_0) = y_0, \;\; y'(t_0) = y_0'$

2nd order LINEAR differential equation:

Thm 3.2.1: If $p : (a, b) \to R$, $q : (a, b) \to R$, and $g : (a, b) \to R$ are continuous and $a < t_0 < b$, then $\exists!$ function $y = \phi(t)$, $\phi : (a, b) \to R$ that satisfies

IVP: $y'' + p(t)y' + q(t)y = g(t), \;\; y(t_0) = y_0, \;\; y'(t_0) = y_0'$

Thm 3.2.2: If ϕ_1 and ϕ_2 are two solutions to a homogeneous linear differential equation

$$y'' + p(t)y' + q(t)y = 0$$

then $c_1\phi_1 + c_2\phi_2$ is also a solution to this linear differential equation.

Proof of thm 3.2.2:

Since $y(t) = \phi_i(t)$ is a solution to the linear homogeneous differential equation $y'' + py' + qy = 0$ where p and q are functions of t (note this includes the case with constant coefficients), then

Claim: $y(t) = c_1\phi_1(t) + c_2\phi_2(t)$ is also a solution to $y'' + py' + qy = 0$

Pf of claim:

Claim: If $y = \phi_1(t)$ and $y = \phi_2(t)$ are linearly independent solutions to $y'' + py' + qy = 0$, then

$$\text{general solution is } y(t) = c_1\phi_1(t) + c_2\phi_2(t)$$

That is any solution to this linear 2nd order homogeneous differential equation can be written as a linear combination of the linear independent functions $y = \phi_1(t)$ and $y = \phi_2(t)$.

Derivation of general solutions:

Solve $ay'' + by' + cy = 0$. Educated guess $y = e^{rt}$, then

$$ar^2e^{rt} + bre^{rt} + ce^{rt} = 0 \text{ implies } ar^2 + br + c = 0,$$

Suppose $r = r_1, r_2$ are solutions to $ar^2 + br + c = 0$

$$r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If $b^2 - 4ac > 0$ we guessed e^{rt} is a solution and noted that any linear combination of solutions is a solution to a homogeneous linear differential equation.

Section 3.3: If $b^2 - 4ac < 0$, :

Changed format of $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ to linear combination of real-valued functions instead of complex valued functions by using Euler's formula:

$$e^{it} = cos(t) + isin(t)$$

Hence $e^{(d+in)t} = e^{dt} e^{int} = e^{dt}[cos(nt) + isin(nt)]$

Let $r_1 = d + in$, $r_2 = d - in$

$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

Section 3.4: If $b^2 - 4ac = 0$, then $r_1 = r_2$.
Hence one solution is $y = e^{r_1 t}$ Need second solution.

If $y = e^{rt}$ is a solution, $y = ce^{rt}$ is a solution.

How about $y = v(t)e^{rt}$?