

3.5: Solving 2nd order linear non-homogeneous DE using method of undetermined coefficients.

Example: Solve  $y'' + 4y = 12t + 8\sin(2t)$ .

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Step 1: Solve homogeneous system,  $y'' + 4y = 0$

Step 2a: Find one solution to  $y'' + 4y = 12t$

Step 2b: Find one solution to  $y'' + 4y = 8\sin(2t)$

To find general solution to non-homogeneous LINEAR differential equation: combine all solutions

Step 3: IF IVP, plug in initial values to find constants.

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Step 1: Solve homogeneous system,  $y'' + 4y = 0$

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$$r^2 + 4 = 0 \Rightarrow r^2 = -4 \Rightarrow r = \pm\sqrt{-4} = 0 \pm 2i$$

Hence homogeneous soln is  $y = c_1 \cos(2t) + c_2 \sin(2t)$

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Step 2a: Find one solution to  $y'' + 4y = 12t$

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Possible guess:  $y = At + B$ . Then  $y' = A$  and  $y'' = 0$ .

Plug in:  $0 + 4(At + B) = 12t \Rightarrow 4At + 4B = 12t + 0$

Thus  $4A = 12$  and  $4B = 0 \Rightarrow A = 3$  and  $B = 0$

Thus  $y = 3t$  is a solution to  $y'' + 4y = 12t$ .

Simpler guess: since there is no  $y'$  term, we didn't need the B term in our guess. We could have guessed  $y = At$  instead for this particular problem (and other analogous problems). If you make similar observations when you do your HW, you can save time when you do comparable problems.

Step 2b: Find one solution to  $y'' + 4y = 8\sin(2t)$

Step 2b: Find one solution to  $y'' + 4y = 8\sin(2t)$

Incorrect guess:  $y = A\sin(2t)$ . Then  $y' = 2A\cos(2t)$  and  $y'' = -4A\sin(2t)$ .

Note: since no  $y'$  term, did not include a  $B\cos(2t)$  term in guess.

Plug in:  $-4A\sin(2t) + 4A\sin(2t) = 8\sin(2t)$ .

$$\text{Thus } 0 = 8\sin(2t).$$

Thus equation has no solution for  $A$ . Hence guess is wrong.

Note this guess is wrong because  $y = \sin(2t)$  is a homogeneous solution. This is why we always solve homogeneous equations first. If a function is a solution to a homogeneous equation, then no constant multiple of that function can be a solution to a non-homogeneous solution since it is a homogeneous solution.

If your normal guess is a homogeneous solution:

Multiply it by  $t$

until it is no longer a homogeneous solution.

Incorrect guess:  $y = At\sin(2t)$ .

Then  $y' = A\sin(2t) + 2At\cos(2t)$  and

$$\begin{aligned}y'' &= 2A\cos(2t) + 2A\cos(2t) - 4At\sin(2t) \\ &= 4A\cos(2t) - 4At\sin(2t).\end{aligned}$$

Plug into  $y'' + 4y = 8\sin(2t)$ :

$$4A\cos(2t) - 4At\sin(2t) + 4At\sin(2t) = 8\sin(2t)$$

But this equation has no solution for  $A$ . Note we need to add a cosine term to our guess so that we can cancel out the cosine term on LHS:

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Better guess:  $y = t[A\sin(2t) + B\cos(2t)]$ .

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Best guess:  $y = Bt\cos(2t)$

Then  $y' = B\cos(2t) - 2Bt\sin(2t)$

$$\begin{aligned}\text{and } y'' &= -2B\sin(2t) - 2B\sin(2t) - 4Bt\cos(2t) \\ &= -4B\sin(2t) - 4Bt\cos(2t)\end{aligned}$$

Plug into  $y'' + 4y = 8\sin(2t)$

$$-4B\sin(2t) - 4Bt\cos(2t) + 4Bt\cos(2t) = 8\sin(2t)$$

$$-4B\sin(2t) = 8\sin(2t) \Rightarrow -4B = 8 \Rightarrow B = -2$$

Thus  $y = -2t\cos(2t)$  is a solution to

$$y'' + 4y = 8\sin(2t)$$

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Note: Guessing wrong is NOT a big deal. You can use your wrong guess to determine a correct guess (though guessing right the first time will save you time).

Recall you are looking for ONE solution to your NON-homogeneous equation.

- If you find an infinite number of solns, choose one.
  - If your guess gives you one solution, use it.
  - If your guess leads to no solutions, than make a different (improved) educated guess.
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To find general solution to non-homogeneous LINEAR differential equation: combine all solutions

$$y = c_1\cos(2t) + c_2\sin(2t) + 3t - 2t\cos(2t)$$