Thm 2.8.1 is translated to origin version of Thm 2.4.2:

Thm 2.8.1: Suppose the functions

$$z = f(t, y) \text{ and } z = \frac{\partial f}{\partial y}(t, y)$$

are continuous for all t in $(-a, a) \times (-c, c)$,

then there exists an interval $(-h, h) \subset (-a, a)$ such that there exists a unique function $y = \phi(t)$ defined on $(-h, h)$ that satisfies the following initial value problem:

$$y' = f(t, y), \quad y(0) = 0.$$

Proof outline (note this is a constructive proof and thus the proof is very useful).

Given: $y' = f(t, y)$, $y(0) = 0$ Eqn (*)

f, $\partial f/\partial y$ continuous $\forall (t, y) \in (-a, a) \times (-b, b)$.

Then $y = \phi(t)$ is a solution to (*) iff

$\phi'(t) = f(t, \phi(t)), \quad \phi(0) = 0$ iff

$\int_0^t \phi'(s)ds = \int_0^t f(s, \phi(s))ds, \quad \phi(0) = 0$ iff

$\phi(t) = \phi(t) - \phi(0) = \int_0^t f(s, \phi(s))ds$

Thus $y = \phi(t)$ is a solution to (*)

iff $\phi(t) = \int_0^t f(s, \phi(s))ds$

Construct ϕ using method of successive approximation – also called Picard's iteration method.

Let $\phi_0(t) = 0$ (or the function of your choice)

Let $\phi_1(t) = \int_0^t f(s, \phi_0(s))ds$

Let $\phi_2(t) = \int_0^t f(s, \phi_1(s))ds$

$\vdots$

Let $\phi_{n+1}(t) = \int_0^t f(s, \phi_n(s))ds$

Let $\phi(t) = \lim_{n\to\infty} \phi_n(t)$

To finish the proof, need to answer the following questions (see book or more advanced class):

1.) Does $\phi_n(t)$ exist for all n?

2.) Does sequence ϕ_n converge?

3.) Is $\phi(t) = \lim_{n\to\infty} \phi_n(t)$ a solution to (*).

4.) Is the solution unique.

Example: $y' = t + 2y$. That is $f(t, y) = t + 2y$

Let $\phi_0(t) = 0$

$$\phi_1(t) = \int_0^t f(s, 0)ds = \int_0^t (s + 2(0))ds$$

$$= \int_0^t s ds = \frac{s^2}{2}\Big|_0^t = \frac{t^2}{2}$$

$$\phi_2(t) = \int_0^t f(s, \phi_1(s))ds = \int_0^t f(s, \tfrac{s^2}{2})ds$$

$$= \int_0^t (s + 2(\tfrac{s^2}{2}))ds = \frac{t^2}{2} + \frac{t^3}{3}$$

$$\phi_3(t) = \int_0^t f(s, \phi_2(s))ds = \int_0^t f(s, \tfrac{s^2}{2} + \tfrac{s^3}{3})ds$$

$$= \int_0^t (s + 2(\tfrac{s^2}{2} + \tfrac{s^3}{3}))ds = \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{6}$$

Example: $y' = t + 2y$. That is $f(t, y) = t + 2y$

$$\phi_3(t) = \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{6}$$

$$\phi_4(t) =$$

Example: $y' = t + 2y$. That is $f(t, y) = t + 2y$

$$\phi_3(t) = \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{6}$$

$$\phi_4(t) = \int_0^t f(s, \phi_3(s))ds$$

$$= \int_0^t f(s, \frac{s^2}{2} + \frac{s^3}{3} + \frac{s^4}{6})ds$$

$$= \int_0^t (s + 2(\frac{s^2}{2} + \frac{s^3}{3} + \frac{s^4}{6}))ds$$

$$= \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{6} + \frac{t^5}{15}$$

Determine formula for ϕ_n:

Note patterns:

$$\int_0^t s ds = \frac{t^2}{2} =$$

$$\int_0^t \frac{s^2}{2} ds = \frac{t^3}{3 \cdot 2} =$$

$$\int_0^t \frac{s^3}{3 \cdot 2} ds = \frac{t^4}{4 \cdot 3 \cdot 2} =$$

$$\int_0^t \frac{s^4}{4 \cdot 3 \cdot 2} ds = \frac{t^5}{5 \cdot 4 \cdot 3 \cdot 2} =$$

Thus look for factorials.

$$\phi_0(t) = 0$$

$$\phi_1(t) = \frac{t^2}{2}$$

$$\phi_2(t) = \frac{t^2}{2} + \frac{t^3}{3}$$

$$\phi_3(t) = \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{6}$$

$$\phi_4(t) = \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{6} + \frac{t^5}{15} = \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{3 \cdot 2} + \frac{t^5}{5 \cdot 3}$$

Thus $\phi_n(t) =$

Claim: Suppose $f(t, y) = t + 2y, \quad \phi_0(t) = 0$

and $\phi_{n+1}(t) = \int_0^t f(s, \phi_n(s))ds$,

then $\phi_n(t) = \displaystyle\sum_{k=2}^{\infty} \frac{2^{k-2}}{k!} t^k$

Proof by induction on n

Will cover this after exam 1.

Note modification in HW 4.

Proving Convergence

Ratio Test

Suppose we have the series $\sum a_n$. Define,

$$L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

Then,

1. if $L < 1$ the series is absolutely convergent (and hence convergent).
2. if $L > 1$ the series is divergent.
3. if $L = 1$ the series may be divergent, conditionally convergent, or absolutely convergent.

https://tutorial.math.lamar.edu/Classes/CalcII/RatioTest.aspx

Prove $\phi_n(t) = \sum_{k=2}^{\infty} \frac{2^{k-2}}{k!} t^k$ converges.

Example: Determine convergence of $\displaystyle\sum_{k=5}^{\infty} \frac{2^k}{3^{k+1}} t^{k-1}$

Defn: $$\sum_{k=0}^{\infty} a_k x^k = \lim_{n\to\infty} \sum_{k=0}^{n} a_k x^k$$
$$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots.$$

Taylor's Theorem: If f is analytic at 0, then for small x (i.e., x near 0),
$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$$
$$= f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 + \ldots$$

Example:

$$e^t = \sum_{k=0}^{\infty} \frac{t^k}{k!} \text{ and thus } e^{bt} = \sum_{k=0}^{\infty} \frac{b^k t^k}{k!} \text{ for } t \text{ near 0.}$$

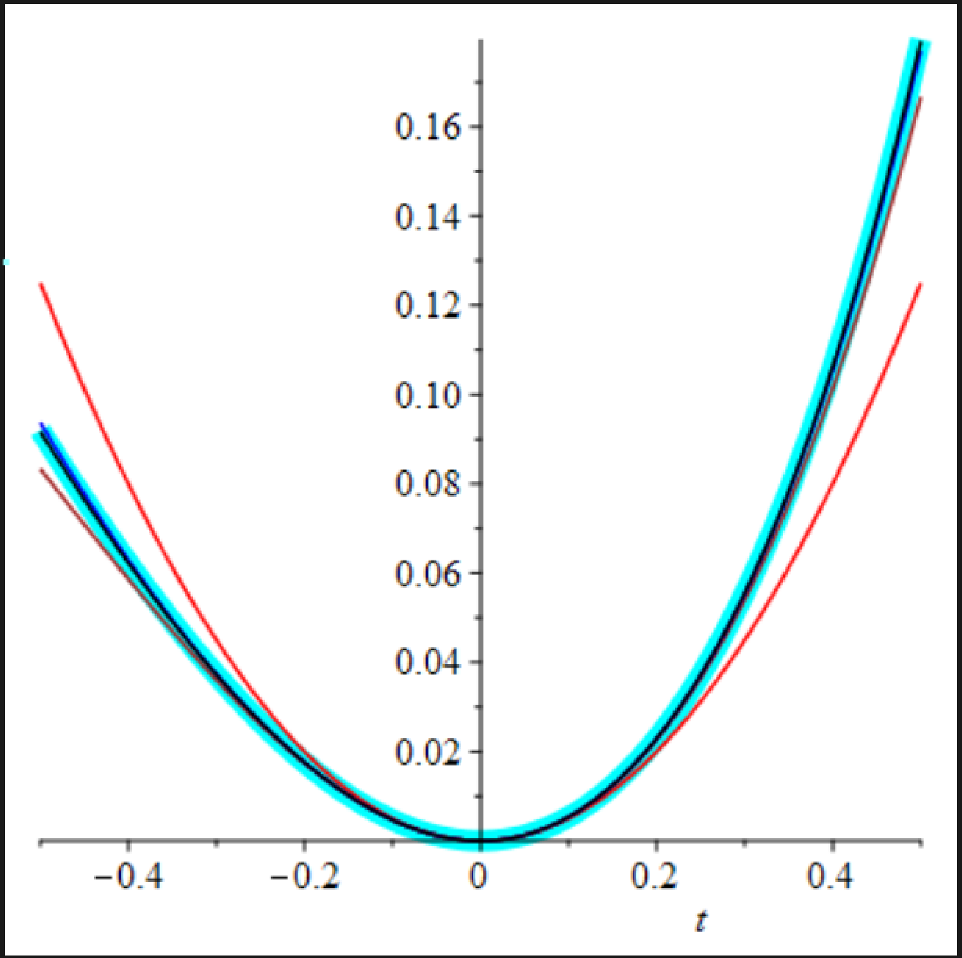
$$\phi_n(t) = \sum_{k=2}^{n} \frac{2^{k-2}}{k!} t^k$$

Thus $\phi(t) = \lim_{n\to\infty} \phi_n(t) = \sum_{k=2}^{\infty} \frac{2^{k-2}}{k!} t^k = \frac{1}{4} \sum_{k=2}^{\infty} \frac{2^k}{k!} t^k$

$$= \frac{1}{4} \left(\qquad - \qquad - \qquad \right)$$

2.8: Approximating soln to IVP using seq of fns.

The solution $y = \phi(t)$ is the thick cyan curve.



$\phi_0(t) = 0$

$\phi_1(t) = \frac{t^2}{2}$

$\phi_2(t) = \frac{t^2}{2} + \frac{t^3}{3}$

$\phi_3(t) = \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{6}$

$\phi_4(t) = \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{6} + \frac{t^5}{15}$