2.4

Special cases:

When do we know a unique solution exists?

Calculus 1 problem: last year

Suppose f is cont. on (a, b) and $t_0 \in (a, b)$,

IVP from Calculus: $\frac{dy}{dt} = f(t)$, $y(t_0) = y_0$

$$dy = \int f(t)\,dt$$

not calc 1

$$\frac{dy}{dt} = y^{\frac{1}{3}}$$

not unique

$y = F(t) + C$ where F is any anti-derivative of F.

Initial Value Problem (IVP): $y(t_0) = y_0$

$$y_0 = F(t_0) + C \text{ implies } C = y_0 - F(t_0)$$

Hence unique sol'n (if domain connected) to IVP:

$$y = F(t) + y_0 - F(t_0)$$

First order linear differential equation:

Thm 2.4.1: If p and g are continuous on (a, b) and the point $t_0 \in (a, b)$, then there exists a unique function $y = \phi(t)$ defined on (a, b) that satisfies the following initial value problem:

$$y' + p(t)y = g(t), \quad y(t_0) = y_0.$$

Prove by deriving the sol'n

Proof: Solve $y' + p(t)y = g(t)$

Let $F(t)$ be an anti-derivative of $p(t)$.

F(t) exists since p(t) is continuous on (a,b)

Thus $p(t) = F'(t)$

Integrating Factor is $u(t) = \underbrace{e^{\int p(t)dt}} = e^{F(t)}$

$[y' + p(t)y = g(t)]e^{F(t)}$

$e^{F(t)}y' + [p(t)e^{F(t)}]y = g(t)e^{F(t)}$

$g(t)e^{F(t)}$ is continuous on (a,b) since g & F are cont

F cont since integral of cont fn on (a,b)

$e^{F(t)}y' + [F'(t)e^{F(t)}]y = g(t)e^{F(t)}$

product rule

FTC

$\int [e^{F(t)}y]'dt = \int g(t)e^{F(t)}dt$

$e^{F(t)} \neq 0$

$e^{F(t)}y = \int g(t)e^{F(t)}dt \Longrightarrow \boxed{y = e^{-F(t)} \int g(t)e^{F(t)}dt}$

Let $A(t)$ be an antiderivative of $g(t)e^{F(t)}$.

Note $A(t)$ exists since $g(t)e^{F(t)}$ is a continuous function.

$$y = e^{-F(t)} \int g(t)e^{F(t)}\,dt = e^{-F(t)}(A(t) + C)$$

Thus general solution is $y = e^{-F(t)}A(t) + Ce^{-F(t)}$

If $y(t_0) = y_0$, then $y_0 = e^{-F(t_0)}A(t_0) + Ce^{-F(t_0)}$

Thus $\boxed{C = e^{F(t_0)}(y_0 - e^{-F(t_0)}A(t_0))}$

$e^{-F(t_0)} \neq 0$

Thus there is a solution for C and that solution is unique.

Hence the IVP $y' + p(t)y = g(t), \;\; y(t_0) = y_0$ has a unique solution on the interval (a, b).

Let $A(t)$ be an antiderivative of $g(t)e^{F(t)}$.

Note $A(t)$ exists on (a, b) since $g(t)e^{F(t)}$ is a continuous function.

$y = e^{-F(t)} \int g(t)e^{F(t)}dt = e^{-F(t)}(A(t) + C)$

Thus general solution is $\boxed{y = e^{-F(t)}A(t) + Ce^{-F(t)}}$

If $y(t_0) = y_0$, then $y_0 = e^{-F(t_0)}A(t_0) + Ce^{-F(t_0)}$

Thus $C = e^{-F(t_0)}(y_0 - e^{-F(t_0)}A(t_0))$

Thus $\exists!$ solution for C.

there exists a unique

Hence $\exists!$ solution defined on (a, b) to the IVP

$$y' + p(t)y = g(t), y(t_0) = y_0.$$

Domain could be larger

1 variable

$p(t)$ & $g(t)$ continuous

Domain

First order linear differential equation:

Thm 2.4.1: If p and g are continuous on (a, b) and the point $t_0 \in (a, b)$, then there exists a unique function $y = \phi(t)$ defined on (a, b) that satisfies the following initial value problem:

where is solution valid

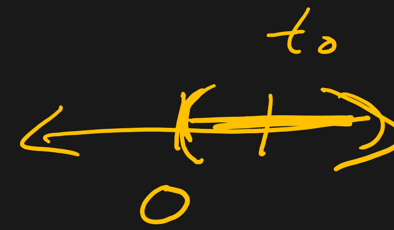
$$\boxed{y' + p(t)y = g(t), \quad y(t_0) = y_0.}$$

$\boxed{\text{Domain}}$

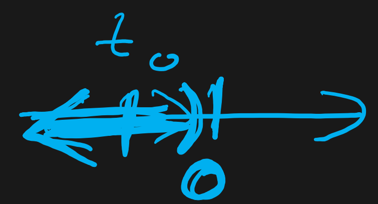
$$\phi : \underbrace{(a, b)}_{\text{domain}} \to \mathbb{R}$$

Example 1: $ty' - y = 1$, $y(t_0) = y_0$

t



$$1y' - \frac{1}{t}y = \frac{1}{t}$$



$$p(t) = -\frac{1}{t}, \quad g(t) = \frac{1}{t}$$

$\Rightarrow p$ & g are continuous for all $t \neq 0$

If $t_0 < 0$, $\exists!\ \phi : (-\infty, 0) \to \mathbb{R}$

If $t_0 > 0$, $\exists!\ \phi : (0, \infty) \to \mathbb{R}$

multivariable fns

More general case (but still need hypothesis)

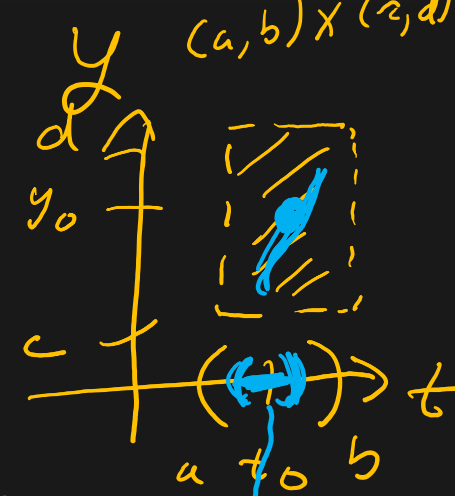
hypothesis

Thm 2.4.2: Suppose the functions

$$z = f(t, y) \text{ and } z = \frac{\partial f}{\partial y}(t, y)$$

are continuous on $(a, b) \times (c, d)$

and the point $(t_0, y_0) \in \underbrace{(a, b)}_{\text{interval}} \times \underbrace{(c, d)}_{\text{interval}}$,



concl

then $\exists$ an interval $(t_0 - h, t_0 + h) \subset (a, b)$ such that $\exists!$ function $y = \phi(t)$ defined on $(t_0 - h, t_0 + h)$ that satisfies the following initial value problem:

don't know what h is, so don't know domain

$$\boxed{y' = f(t, y), \quad y(t_0) = y_0.}$$

← IVP 1rst order DE

If possible **without solving**, determine where the solution exists for the following initial value problems:

linear

domain

If not possible **without solving**, state where in the ty-plane, the hypothesis of theorem 2.4.2 is satisfied. In other words, use theorm 2.4.2 to determine where for some rectangle about the point (t_0, y_0), a solution to IVP, $y' = f(t, y)$, $y(t_0) = y_0$ exists and is unique.

Thm 2.4.1 stronger

Example 1: $ty' - y = 1, \; y(t_0) = y_0$

Thm 2.4.2

Example 2: $y' = ln|\frac{t}{y}|, \; y(3) = 6$ ← not linear

Example 3: $(t^2 - 1)y' - \frac{t^3 y}{t-4} = ln|t|, \; y(3) = 6$

Thm 2.4.1

Example 2: $y' = ln|\frac{t}{y}|$, $y(3) = 6$

Not linear

must use

$\Rightarrow$ Thm 2.4.2

or solve

hyp

Thm 2.4.2: Suppose the functions

$$z = f(t, y) \text{ and } z = \frac{\partial f}{\partial y}(t, y)$$

are continuous on $(a, b) \times (c, d)$

and the point $(t_0, y_0) \in (a, b) \times (c, d)$,

hyp

then $\exists$ an interval $(t_0 - h, t_0 + h) \subset (a, b)$ such that $\exists!$ function $y = \phi(t)$ defined on $(t_0 - h, t_0 + h)$ that satisfies the following initial value problem:

$$y' = f(t, y), \quad y(t_0) = y_0.$$

Example 2: $y' = ln|\frac{t}{y}|, y(3) = 6$

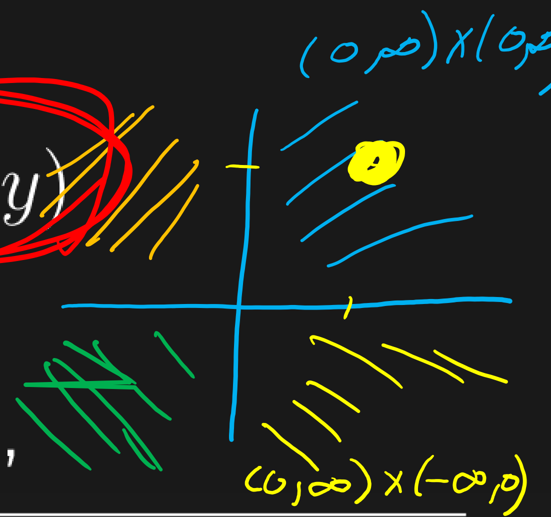
$f(t, y) = \ln|\frac{t}{y}|$

Thm 2.4.2: Suppose the functions

$$z = f(t, y) \text{ and } z = \frac{\partial f}{\partial y}(t, y)$$

are continuous on $(a, b) \times (c, d)$

and the point $(t_0, y_0) \in (a, b) \times (c, d)$,



Example 2: $y' = ln|\frac{t}{y}|$, $y(3) = 6$

$(3, 6)$

$f(t, y) = \ln\left|\frac{t}{y}\right|$ is continues

if $t \neq 0$, $y \neq 0$

$(3, 6) \in (0, \infty) \times (0, \infty)$

Thm 2.4.2: Suppose the functions

$$z = f(t, y) \text{ and } z = \frac{\partial f}{\partial y}(t, y)$$

are continuous on $(a, b) \times (c, d)$

and the point $(t_0, y_0) \in (a, b) \times (c, d)$,

hyp

Example 2: $y' = ln|\frac{t}{y}|$, $y(3) = 6$

f & $\frac{\partial f}{\partial y}$ are cont $\forall t \neq 0$ $\forall y \neq 0$

$$\frac{\partial}{\partial y}\left(\ln|ty^{-1}|\right) = \frac{1}{t/y}\left(-ty^{-2}\right)$$

$$= \frac{y}{\cancel{t}} \cdot \frac{-\cancel{t}}{y^2} = -\frac{1}{y}$$

cont $\forall t$ $\forall y \neq 0$

Thm 2.4.2:

then $\exists$ an interval $(t_0 - h, t_0 + h) \subset (a, b)$ such that $\exists!$ function $y = \phi(t)$ defined on $(t_0 - h, t_0 + h)$ that satisfies the following initial value problem:

$$y' = f(t, y), \quad y(t_0) = y_0.$$

Example 2: $y' = ln|\frac{t}{y}|$, $y(3) = 6$

$3 \neq 0$, $6 \neq 0$ $\quad (3,6) \in (0,\infty) \times (0,\infty)$

& f & $\frac{\partial f}{\partial y}$ are cont on $(0,\infty) \times (0,\infty)$

$\Rightarrow$ IVP has a unique soln

but we do NOT know domain

Example 3: $(t^2-1)y' - \frac{t^3 y}{t-4} = ln|t|,\ y(3)=6$

Use thm 2.4.1 : hyp is easier and conclusion is stronger

$$1y' - \frac{t^3 y}{(t-4)(t^2-1)} = \frac{ln|t|}{t^2-1}$$

$$p(t) = \frac{t^3}{(t-4)(t-1)(t+1)}$$ is cont if $t \neq \underline{4}, \underline{1}, \underline{-1}$

$$g(t) = \frac{ln|t|}{(t-1)(t+1)}$$ is cont if $t \neq \underline{0}, \underline{1}, \underline{-1}$

First order linear differential equation:

Thm 2.4.1: If p and g are continuous on (a, b) and the point $t_0 \in (a, b)$, then there exists a unique function $y = \phi(t)$ defined on (a, b) that satisfies the following initial value problem:

$$y' + p(t)y = g(t), \quad y(t_0) = y_0.$$

Example 3: $\frac{(t^2-1)y'}{t^2-1} - \frac{t^3y}{(t^2-1)(t-4)} = \frac{\ln|t|}{(t^2-1)}, \quad y(3) = 6$

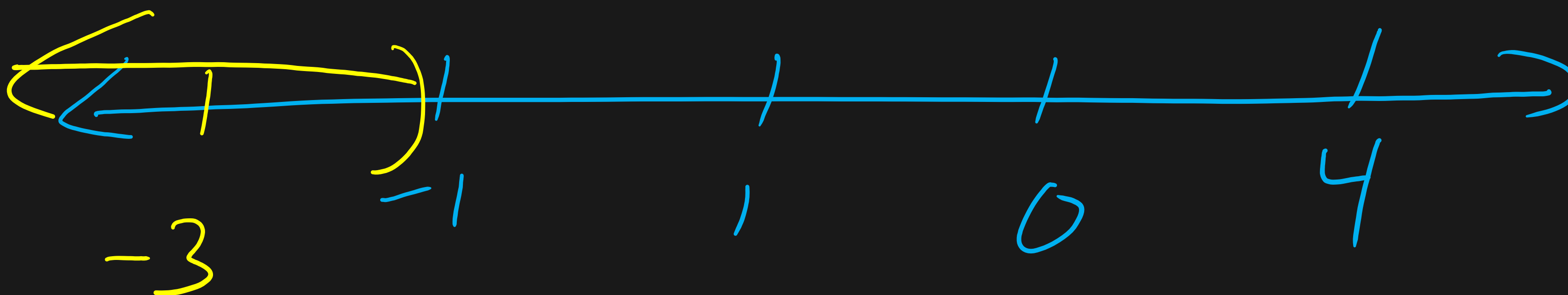
$p(t)$ & $g(t)$ are cont if

$t \neq -1, 0, 1, 4$

$\exists!$ soln $\phi : (1, 4) \to \mathbb{R}$

Example 3: $(t^2 - 1)y' - \frac{t^3 y}{t-4} = ln|t|$, $\cancel{y(3) = 6}$

$y(-3) =$ [illegible]



$\exists! \ \phi : (-\infty, -1) \to \mathbb{R}$

In each of Problems 1 through 4, determine (without solving the problem) an interval in which the solution of the given initial value problem is certain to exist.

1. $(t-3)\,y' + (\ln t)\,y = 2t, \quad y(1) = 2$
2. $y' + (\tan t)\,y = \sin t, \quad y(\pi) = 0$
3. $(4-t^2)\,y' + 2ty = 3t^2, \quad y(-3) = 1$
4. $(\ln t)\,y' + y = \cot t, \quad y(2) = 3$

In each of Problems 5 through 8, state where in the ty-plane the hypotheses of Theorem 2.4.2 are satisfied.

5. $y' = (1-t^2-y^2)^{1/2}$
6. $y' = \dfrac{\ln|ty|}{1-t^2+y^2}$
7. $y' = (t^2+y^2)^{3/2}$