Quickly written notes:

2.2: example from week 3 video (slides 12 - 14) available at
http://homepage.math.uiowa.edu/∼ idarcy/COURSES/100/SPRING22/videos.html

2.3: Modeling with differential equations.

Suppose salty water enters and leaves a tank at a rate of 6 liters/minute.

Suppose also that the salt concentration of the water entering the tank varies with respect to time according to $\frac{1}{(t+1)(t+2)e^{3t}}$ g/liters where $Q(t)$ = amount of salt in tank in grams.

If the tank contains 2 liters of water and initially contains 1g of salt, find a formula for the amount of salt in the tank after t minutes.

Let $Q(t)$ = amount of salt in tank in grams. Note $Q(0) = 1$ g

rate in = (6 liters/min)($\frac{1}{(t+1)(t+2)e^{3t}}$ g/liters) = $\frac{6}{(t+1)(t+2)e^{3t}}$ g/min

rate out = (6 liters/min)($\frac{Q(t)g}{2 \text{ liters}}$) = $3Q$ g/min

$\frac{dQ}{dt}$ = rate in - rate out = $\frac{6}{(t+1)(t+2)e^{3t}} - 3Q$

IVP: $\frac{dQ}{dt} = \frac{6}{(t+1)(t+2)e^{3t}} - 3Q, \quad Q(0) = 1$

$Q' = \frac{6}{(t+1)(t+2)e^{3t}} - 3Q$

$Q' + 3Q = \frac{6}{(t+1)(t+2)e^{3t}}$

$u(t) = e^{\int 3dt} = e^{3t}$

$e^{3t}Q' + 3e^{3t}Q = \frac{6e^{3t}}{e^{3t}(t+1)(t+2)}$

$(e^{3t}Q)' = \frac{6}{(t+1)(t+2)}$

$\int (e^{3t}Q)'dt = \int \frac{6}{(t+1)(t+2)}dt$

$e^{3t}Q = \int \frac{6}{(t+1)(t+2)}dt$

Partial fractions: $\frac{6}{(t+1)(t+2)} = \frac{A}{t+1} + \frac{B}{t+2}$

$6 = A(t+2) + B(t+1)$

Method 1: choose 2 values for t to solve for the 2 variables.

$t = -1$: $6 = A$ $\qquad\qquad$ $t = -2$: $6 = -B$

Method 2: use linear independence.

$0t + 6 = (A + B)t + 2A + B$

$A + B = 0$, thus $B = -A$

$2A + B = 6$. Thus $2A - A = A = 6$ and $B = -6$

Hence $\frac{6}{(t+1)(t+2)} = \frac{6}{t+1} + \frac{-6}{t+2}$

$e^{3t}Q = \int \frac{6}{(t+1)(t+2)} dt = \int \frac{6}{t+1} dt + \int \frac{-6}{t+2} dt$

$e^{3t}Q = 6ln|t + 1| - 6ln|t + 2| + C$

$Q = 6e^{-3t}ln|t + 1| - 6e^{-3t}ln|t + 2| + Ce^{-3t}$

Initial value: $Q(0) = 1$

$1 = 6ln|1| - 6ln|2| + C$

$C = 1 + 6ln(2)$

$Q = 6e^{-3t}ln|t + 1| - 6e^{-3t}ln|t + 2| + (1 + 6ln(2))e^{-3t}$

Long-term behaviour:

As $t \to \infty$, $6e^{-3t}ln|t + 1| - 6e^{-3t}ln|t + 2| + (1 + 6ln(2))e^{-3t} \to 0$

You know the limit is 0 using any of the 3 methods below:

1.) Note the salt concentration entering the tank goes to 0 as $t \to +\infty$

2.) Note that the exponential function goes to 0 much faster than the logarithm function goes to $+\infty$ as $t \to +\infty$

3.) Use l'hopital's rule. For example,

$\lim_{t\to\infty} 6e^{-3t}ln|t + 1|$

$$\lim_{t\to\infty} 6\frac{ln|t + 1|}{e^{3t}} = \lim_{t\to\infty} 6\frac{\frac{1}{t+1}}{3e^{3t}} = \lim_{t\to\infty} \frac{6}{3(t + 1)e^{3t}} = 0$$