3.5: Non-homogeneous linear differential equation

$$a(t)y'' + b(t)y' + c(t)y = g(t) \qquad \text{where } g(t) \neq 0$$

Step 1: Find homogeneous general solution to $a(t)y'' + b(t)y' + c(t)y = 0$:

$$y = c_1\phi_1(t) + c_2\phi_2(t)$$

Step 2: Find one non-homogeneous solution to $a(t)y''+b(t)y'+c(t)y = g(t)$: ψ

Step 3: Combine these solutions to create general solution to the non-homogeneous linear DE:

plug in

$$y = \underbrace{c_1\phi_1(t) + c_2\phi_2(t)}_{0} + \underbrace{\psi}_{g(t)} = g(t)$$

Compare to linear algebra: non-homogenous solution is a "shifted" version of the homogeneous solution (suppose solution space is 2-dimensional):

LINEAR EQN

$$A\mathbf{x} = \mathbf{0} \quad \Rightarrow \quad \mathbf{x} = c_1\mathbf{v_1} + c_2\mathbf{v_2}$$
$$A\mathbf{x} = \mathbf{b} \quad \Rightarrow \quad \mathbf{x} = c_1\mathbf{v_1} + c_2\mathbf{v_2} + \mathbf{w}$$

homog & nonhomog + shift

$$a(t)y'' + b(t)y' + c(t)y = 0 \quad \Rightarrow \quad y = c_1\phi_1(t) + c_2\phi_2(t)$$
$$a(t)y'' + b(t)y' + c(t)y = g(t) \quad \Rightarrow \quad y = \underbrace{c_1\phi_1(t) + c_2\phi_2(t)}_{0} + \underbrace{\psi}_{w} \quad + g(t) = g(t)$$

Example 1: Solve $y'' - 4y' + 4y = e^{3t}$

Step 1: Solve homogeneous version:

$$y'' - 4y' + 4y = 0 \quad \Rightarrow \quad r^2 - 4r + 4 = 0 \quad \Rightarrow \quad (r-2)^2 = 0 \quad \Rightarrow \quad r = 2, 2$$

Thus general solution to homogeneous DE is $y = c_1e^{2t} + c_2te^{2t}$

Step 2: Find ONE non-homogeneous solution to $y'' - 4y' + 4y = e^{3t}$

3.5 Method of undetermined coefficients – Educated guess: $y = Ae^{3t}$

Plug in to solve for undetermined coefficient A:

$$y = Ae^{3t} \quad \Rightarrow \quad y' = 3Ae^{3t} \quad \Rightarrow \quad y'' = 9Ae^{3t}$$

$$9Ae^{3t} - 4(3Ae^{3t}) + 4Ae^{3t} = e^{3t} \quad \Rightarrow \quad 9A - 12A + 4A = 1 \quad \Rightarrow \quad A = 1$$

Thus ONE non-homogeneous solution is $y = e^{3t}$

Step 3: Combine these solutions to create general solution to the non-homogeneous linear DE:

The general solution to nonhomogeneous DE is $y = c_1e^{2t} + c_2te^{2t} + e^{3t}$

Example 2: Solve $y'' - 4y' + 4y = 2cos(3t)$ $\Rightarrow$ $Y(t) = A\cos(3t) + B\sin(3t)$

Step 1: Solve homogeneous version:

$y'' - 4y' + 4y = 0 \quad \Rightarrow \quad r^2 - 4r + 4 = 0 \quad \Rightarrow \quad (r-2)^2 = 0 \quad \Rightarrow \quad r = 2, 2$

Thus general solution to homogeneous DE is $y = c_1 e^{2t} + c_2 t e^{2t}$

Step 2: Find ONE non-homogeneous solution to $\boxed{y'' - 4y' + 4y = 2cos(3t)}$

3.5 Method of undetermined coefficients – Educated guess: $y = Acos(3t) + Bsin(3t)$

Plug in to solve for undetermined coefficient A: $y = Acos(3t) + Bsin(3t)$
$\Rightarrow \quad y' = -3Asin(3t) + 3Bcos(3t) \quad \Rightarrow \quad y'' = -9Acos(3t) - 9Bsin(3t)$

plug in and solve for A & B

$$\begin{aligned} y'' &= -9Acos(3t) - 9Bsin(3t) \\ -4y' &= -12Bcos(3t) + 12Asin(3t) \\ +4y &= +4Acos(3t) + 4Bsin(3t) \end{aligned}$$
– Note switched order of B and A term

2 unknowns 2 eqns

$y'' - 4y' + 4y = (-9A - 12B + 4A)cos(3t) + (-9B + 12A + 4B)sin(3t).$

$y'' - 4y' + 4y = (-5A - 12B)cos(3t) + (12A - 5B)sin(3t)$

$y'' - 4y' + 4y = 2cos(3t)$

$\boxed{(-5A - 12B)cos(3t)} + (12A - 5B)sin(3t) = \boxed{2cos(3t)}$ $+ 0\sin(3t)$

$\boxed{-5A - 12B = 2 \text{ and } 12A - 5B = 0}$

From 2nd equation: $5B = 12A$ and $B = \frac{12A}{5}$

Plug into first equation: $\boxed{-5A - 12(\frac{12A}{5}) = 2}$

$-25A - 144A = 10 \quad \Rightarrow \quad -169A = 10 \quad \Rightarrow \quad A = -\frac{10}{169}$

Hence $B = (\frac{12}{5})A = B = (\frac{12}{5})(-\frac{10}{169}) = -(\frac{12}{1})(\frac{2}{169}) = -\frac{24}{169}$

Thus ONE non-homogeneous solution is $y = -\frac{10}{169}cos(3t) - \frac{24}{169}sin(3t)$

Step 3: Combine these solutions to create general solution to the non-homogeneous linear DE:

The general solution to nonhomogeneous DE is

plug in 2cos(3t)

$$\boxed{y = c_1 e^{2t} + c_2 t e^{2t} + -\frac{10}{169}cos(3t) - \frac{24}{169}sin(3t)}$$

general (non hom) soln

Give the form of the particular (nonhomogenous) solution with undetermined coefficients for

$$y'' + 4y' + 4y = 1sin(3t)$$

$y = Y(t) = Y(t) =$ $A\cos(3t) + B\sin(3t)$ (Do NOT solve!)

Give the form of the particular (nonhomogenous) solution with undetermined coefficients for

$$y'' + 4y' + 4y = 4sin(3t)$$

$Y(t) =$ $A\cos(3t) + B\sin(3t)$ (Do NOT solve!)

Give the form of the particular (nonhomogenous) solution with undetermined coefficients for

$$y'' + 4y' + 4y = sin(3t) - 8cos(3t)$$

$Y(t) =$ $A\cos(3t) + B\sin(3t)$ (Do NOT solve!)

Answers:

Give the form of the particular (nonhomogenous) solution with undetermined coefficients for

$$y'' + 4y' + 4y = sin(3t)$$

$Y(t) =$ $y = Acos(3t) + Bsin(3t)$ (Do NOT solve!)

Give the form of the particular (nonhomogenous) solution with undetermined coefficients for

$$y'' + 4y' + 4y = 4sin(3t)$$

$Y(t) =$ $y = Acos(3t) + Bsin(3t)$ (Do NOT solve!)

Give the form of the particular (nonhomogenous) solution with undetermined coefficients for

$$y'' + 4y' + 4y = sin(3t) - 8cos(3t)$$

$Y(t) =$ $y = Acos(3t) + Bsin(3t)$ (Do NOT solve!)

To solve linear DE $ay'' + by' + cy = g_1 + g_2 + g_3$.

Step 1: Solve homogeneous version: $ay'' + by' + cy = 0$ implies
$ar^2 + br + c = 0$ implies ... $\boxed{y = c_1\phi_1 + c_2\phi_2}$.

Step 2a: Find one non-homogeneous solution, $y = f_1$, to $ay'' + by' + cy = g_1$
Step 2b: Find one non-homogeneous solution, $y = f_2$, to $ay'' + by' + cy = g_2$
Step 2c: Find one non-homogeneous solution, $y = f_3$, to $ay'' + by' + cy = g_3$

Step 3: Combine all solutions to create the general solution to the non-homogeneous DE:

$$y = \underbrace{c_1\phi_1 + c_2\phi_2}_{0} + f_1 + f_2 + f_3$$

plug in

$0 + g_1 + g_2 + g_3$

Last step: If IVP, plug in initial values to find the constants c_1 and c_2.

Guess a possible non-homog soln for the following DEs:

Note homogeneous solution to $y'' + 2y' + y = 0$ is
$y = c_1e^{-t} + c_2te^{-t}$ since $r^2 + 2r + 1 = (r+1)(r+1) = 0$

1.) $y'' + 2y' + y = 4e^{2t}$

Guess: $y = Ae^{2t}$

2.) $y'' + 2y' + y = 4e^{t}$

Guess: $y = Ae^{t}$

Remember product rule

3.) $y'' + 2y' + y = 4e^{-t}$

Guess: $y = At^2e^{-t}$

$\underbrace{(Ate^{-t})'' + 2(Ate^{-t})' + Ate^{-t}}$ $= 0$

plug in

$y = Ae^{-t}$ is a homog soln
$\Rightarrow$ multiply by t
But $y = Ate^{-t}$ is also a homog soln
$\Rightarrow$ multiply by t again (until no longer homog)

4.) $y'' + 2y' + y = t$ $y = At$ $\Rightarrow y' = A$ constant term $[2A + At = t]$ no soln need to add + constant

Guess: $y = At + B$

5.) $y'' + 2y' + y = [t + 1]$ $g(t) =$ degree 1 polynom $=$ guess

Guess: $y = At + B$ plug in $[2A + At + B = t + 1]$

6.) $y'' + 2y' + y = 4sin(2t)$

Guess: $y = A\cos(2t) + B\sin(2t)$

7.) $y'' + 2y' + y = [4sin(2t) + 5cos(2t)]$

Guess: $y = A\cos(2t) + B\sin(2t)$

8.) $y'' + 2y' + y = [4sin(2t)] + [5cos(3t)]$

Guess for step 2a: $y = A_1\cos(2t) + B_1\sin(2t)$

Guess for step 2b: $y = A_2\cos(3t) + B_2\sin(3t)$

9.) $y'' + 2y' + y = [4sin(2t)] + [t + 1]$

Guess for step 2a: $y = A\cos(2t) + B\sin(2t)$

Guess for step 2b: $y = Ct + D$

10.) $y'' + 2y' + y = 4tsin(2t)$ $y = (At + B)(C\cos(2t) + D\sin(2t))$

Guess: $y = A_1 t\cos(2t) + A_2\cos(2t) + A_3 t\sin(2t) + A_4\sin(2t)$

11.) $y'' + y = 4sin(2t)$

Guess: $y = A\sin(2t)$ don't need the cos term since no y' term

but $y = A\cos(2t) + B\sin(2t)$ is also correct

12.) $y'' + y = 4sin(t)$

Guess:

3.5: Solving non-homogeneous linear DE using the undetermined coefficients method

1.) Step 1: Solve homogeneous version of DE.

2.) Step 2: Guess a non-homogeneous solution with undetermined coefficients. Plug into the non-homogeneous linear DE to solve for the undetermined coefficients.

3.) Combing general homogeneous solution with a non-homogeneous solution.

Starting guess:

If $ay'' + by' + cy = ke^{pt}$, guess $y = Ae^{pt}$

If $ay'' + by' + cy = ksin(pt) + jcos(pt)$, guess $y = Asin(pt) + Bcos(pt)$

If $ay'' + by' + cy =$ degree n polynomial,
guess $y =$ a degree n polynomial including all terms
(with undetermined coefficients) including constant term.

If $ay'' + by' + cy =$ a sum, guess a sum (but usually solve separately).

If $ay'' + by' + cy =$ a product, guess a product.

Sometimes the above can be simplified:

If a term does not show up when you take the derivatives of y, you may be able to omit that term. E.g, $y'' + w^2y = sin(pt)$ where $p \neq w$, then $y = Asin(pt)$ is a simpler guess that works.

If the above does not work

Try multiplying non-simplified guess by t.

Example: If guess is a homogeneous solution, then that will not be a non-homogeneous solution. Thus must guess something else. Multiplying non-simplified guess by t until no longer homogeneous works.

Example: If y term missing, and $g(t) =$ degree n polynomial, then will need to multiply by t so that when you plug in guess, you will have a degree n polynomial on both sides of equal sign.

Note: you are multiplying the **non-simplified guess** by t. When you take derivatives of y, you must use the **product** rule. Thus extra terms appear when you take the derivative and you will need the non simplified guess to