3.5: Non-homogeneous linear differential equation ✓ non homog

$$a(t)y'' + b(t)y' + c(t)y = g(t) \qquad \text{where } g(t) \neq 0$$

Step 1: Find homogeneous general solution to $a(t)y'' + b(t)y' + c(t)y = 0$:

$$y = c_1\phi_1(t) + c_2\phi_2(t)$$

Step 2: Find one non-homogeneous solution to $a(t)y''+b(t)y'+c(t)y = g(t)$:

Step 3: Combine these solutions to create general solution to the non-homogeneous linear DE:

$$y = c_1\phi_1(t) + c_2\phi_2(t) + \psi$$

Compare to linear algebra: non-homogenous solution is a "shifted" version of the homogeneous solution (suppose solution space is 2-dimensional):

$$\begin{aligned} A\mathbf{x} = \mathbf{0} \quad &\Rightarrow \quad \mathbf{x} = c_1\mathbf{v_1} + c_2\mathbf{v_2} \\ A\mathbf{x} = \mathbf{b} \quad &\Rightarrow \quad \mathbf{x} = c_1\mathbf{v_1} + c_2\mathbf{v_2} + \mathbf{w} \end{aligned}$$

shift homog soln to get non homog soln

$$\begin{aligned} a(t)y'' + b(t)y' + c(t)y = 0 \quad &\Rightarrow \quad y = c_1\phi_1(t) + c_2\phi_2(t) \\ a(t)y'' + b(t)y' + c(t)y = g(t) \quad &\Rightarrow \quad y = c_1\phi_1(t) + c_2\phi_2(t) + \psi \end{aligned}$$

Example 1: Solve $y'' - 4y' + 4y = e^{3t}$

plug in

$$0 + g = g$$

Step 1: Solve homogeneous version:

$$y'' - 4y' + 4y = 0 \quad \Rightarrow \quad r^2 - 4r + 4 = 0 \quad \Rightarrow \quad (r-2)^2 = 0 \quad \Rightarrow \quad r = 2, 2$$

Thus general solution to homogeneous DE is $y = c_1e^{2t} + c_2te^{2t}$

Step 2: Find ONE non-homogeneous solution to $y'' - 4y' + 4y = e^{3t}$

3.5 Method of undetermined coefficients – Educated guess: $y = Ae^{3t}$

Plug in to solve for undetermined coefficient A:

$$y = Ae^{3t} \quad \Rightarrow \quad y' = 3Ae^{3t} \quad \Rightarrow \quad y'' = 9Ae^{3t}$$

$$9Ae^{3t} - 4(3Ae^{3t}) + 4Ae^{3t} = e^{3t} \quad \Rightarrow \quad 9A - 12A + 4A = 1 \quad \Rightarrow \quad A = 1$$

Thus ONE non-homogeneous solution is $y = e^{3t}$

Step 3: Combine these solutions to create general solution to the non-homogeneous linear DE:

The general solution to nonhomogeneous DE is $y = c_1e^{2t} + c_2te^{2t} + e^{3t}$

Example 2: Solve $y'' - 4y' + 4y = 2cos(3t)$

Step 1: Solve homogeneous version:

$y'' - 4y' + 4y = 0 \quad \Rightarrow \quad r^2 - 4r + 4 = 0 \quad \Rightarrow \quad (r-2)^2 = 0 \quad \Rightarrow \quad r = 2, 2$

Thus general solution to homogeneous DE is $y = c_1 e^{2t} + c_2 te^{2t}$

Step 2: Find ONE non-homogeneous solution to $y'' - 4y' + 4y = 2cos(3t)$

3.5 Method of undetermined coefficients – Educated guess: $y = Acos(3t) + Bsin(3t)$

Plug in to solve for undetermined coefficient A: $y = Acos(3t) + Bsin(3t)$
$\Rightarrow \quad y' = -3Asin(3t) + 3Bcos(3t) \quad \Rightarrow \quad y'' = -9Acos(3t) - 9Bsin(3t)$

$$\begin{aligned} y'' &= \quad -9Acos(3t) - 9Bsin(3t) \\ -4y' &= -12Bcos(3t) + 12Asin(3t) \qquad \text{– Note switched order of } B \text{ and } A \text{ term} \\ +4y &= \quad +4Acos(3t) + 4Bsin(3t) \end{aligned}$$

$y'' - 4y' + 4y = (-9A - 12B + 4A)cos(3t) + (-9B + 12A + 4B)sin(3t)$

$y'' - 4y' + 4y = (-5A - 12B)cos(3t) + (12A - 5B)sin(3t)$

$y'' - 4y' + 4y = 2cos(3t)$

$\boxed{(-5A - 12B)cos(3t) + (12A - 5B)sin(3t) = 2cos(3t)}$ + 0sin(3t)

$-5A - 12B = 2$ and $12A - 5B = 0$

2 eqns w/ 2 unknowns

plug it into DE and solve for undetermined coeff A & B

From 2nd equation: $5B = 12A$ and $B = \frac{12A}{5}$

Plug into first equation: $-5A - 12(\frac{12A}{5}) = 2$

$-25A - 144A = 10 \quad \Rightarrow \quad -169A = 10 \quad \Rightarrow \quad A = -\frac{10}{169}$

Hence $B = (\frac{12}{5})A = B = (\frac{12}{5})(-\frac{10}{169}) = -(\frac{12}{1})(\frac{2}{169}) = -\frac{24}{169}$

Thus ONE non-homogeneous solution is $y = -\frac{10}{169}cos(3t) - \frac{24}{169}sin(3t)$

Step 3: Combine these solutions to create general solution to the non-homogeneous linear DE:

The general solution to nonhomogeneous DE is

$$y = c_1 e^{2t} + c_2 te^{2t} + -\frac{10}{169}cos(3t) - \frac{24}{169}sin(3t)$$

~~to DE unique soln by Thm in sect 3.2~~

Give the form of the particular (nonhomogenous) solution with undetermined coefficients for

$$y'' + 4y' + 4y = sin(3t)$$

$Y(t) =$ $A\cos(3t) + B\sin(3t)$ (Do NOT solve!)

Give the form of the particular (nonhomogenous) solution with undetermined coefficients for

$$y'' + 4y' + 4y = 4sin(3t)$$

$Y(t) =$ $A\cos(3t) + B\sin(3t)$ (Do NOT solve!)

Give the form of the particular (nonhomogenous) solution with undetermined coefficients for

$$y'' + 4y' + 4y = sin(3t) - 8cos(3t)$$

$Y(t) =$ $A\cos(3t) + B\sin(3t)$ (Do NOT solve!)

Answers:

Give the form of the particular (nonhomogenous) solution with undetermined coefficients for

$$y'' + 4y' + 4y = sin(3t)$$

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To solve linear DE $ay'' + by' + cy = g_1 + g_2 + g_3$

Step 1: Solve homogeneous version: $ay'' + by' + cy = 0$ implies
$ar^2 + br + c = 0$ implies $\quad \ldots \quad y = c_1\phi_1 + c_2\phi_2$.

Step 2a: Find one non-homogeneous solution, $y = f_1$, to $ay'' + by' + cy = g_1$
Step 2b: Find one non-homogeneous solution, $y = f_2$, to $ay'' + by' + cy = g_2$
Step 2c: Find one non-homogeneous solution, $y = f_3$, to $ay'' + by' + cy = g_3$

Step 3: Combine all solutions to create the general solution to the non-homogeneous DE:

$$y = \underbrace{c_1\phi_1 + c_2\phi_2}_{} + f_1 + f_2 + f_3$$

plug it in $\Rightarrow$ $0 \quad + \quad g_1 + g_2 + g_3$

Last step: If IVP, plug in initial values to find the constants c_1 and c_2.

Guess a possible non-homog soln for the following DEs:

Note homogeneous solution to $y'' + 2y' + y = 0$ is
$y = c_1 e^{-t} + c_2 t e^{-t}$ since $r^2 + 2r + 1 = (r+1)(r+1) = 0$

1.) $y'' + 2y' + y = 4e^{2t}$

Guess: $y = Ae^{2t}$

2.) $y'' + 2y' + y = 4e^t$

Guess: $y = Ae^{t}$

remember product rule and to cancel out terms that give 0

ex $[A(e^{-t})'' + 2(e^{-t})' + e^{-t}] = 0$ LHS of DE

$y = Ate^{-t}$ is also homog soln so multiply by t

3.) $y'' + 2y' + y = 4e^{-t}$

Guess: $y = At^2 \cdot e^{-t}$

since homog sol'n multiply by t until it is no longer a homog soln

e^{-t} is a homog soln so if you plug in Ae^{-t}, you will get 0

4.) $y'' + 2y' + y = t$

Guess: $y = At + B$

since $(At)' = A$ will need constant B to cancel out constant term

5.) $y'' + 2y' + y = t + 1$

Guess: $y = At + B$

6.) $y'' + 2y' + y = 4sin(2t)$

Guess: $y = A\cos(2t) + B\sin(2t)$

7.) $y'' + 2y' + y = [4sin(2t) + 5cos(2t)]$

Guess: $y = A\cos(2t) + B\sin(2t)$

8.) $y'' + 2y' + y = [\underbrace{4sin(2t)}_{2a}] + [5cos(3t)]$

Guess for step 2a: $y = A_1\cos(2t) + B_1\sin(2t)$

Guess for step 2b: $y = A_2\cos(3t) + B_2\sin(3t)$

and then combine sols: +

9.) $y'' + 2y' + y = [4sin(2t)] + [t + 1]$

Guess for step 2a: $y = A\cos(2t) + B\sin(2t)$

Guess for step 2b: $y = Ct + D$

10.) $y'' + 2y' + y = 4t\,sin(2t) \Rightarrow y = (At + B) \cdot (C\cos(2t) + D\sin(2t))$

order does not matter

Guess: $y = A_1 t\cos(2t) + A_2\cos(2t) + A_3 t\sin(2t) + A_4\sin(2t)$

11.) $y'' + y = 4sin(2t)$

Guess: $y = A\sin(2t)$ ← no y' term so don't need $+B\cos(2t)$

12.) $y'' + y = 4sin(t)$ → $y'' + y = 0 \Rightarrow r^2 + 1 = 0 \Rightarrow r = \pm i \Rightarrow$ homog soln $y = c_1\cos t + c_2\sin t$

homog

Guess: $y = At\sin(t) + Bt\cos t$

↳ need cos term due to product rule

$$y'' + y = t^3 \Rightarrow y = At^3 + Bt^2 + Ct + D$$

$$y'' + y' = t^3 \quad = y = \left[At^3 + Bt^2 + Ct\right] t$$

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$$\begin{aligned} a(t)y'' + b(t)y' + c(t)y = 0 \quad &\Rightarrow \quad y = c_1\phi_1(t) + c_2\phi_2(t) \\ a(t)y'' + b(t)y' + c(t)y = g(t) \quad &\Rightarrow \quad y = c_1\phi_1(t) + c_2\phi_2(t) + \psi \end{aligned}$$

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