7.5 Two real eigenvalues (Example 1: One **positive** and one **negative** eigenvalue).

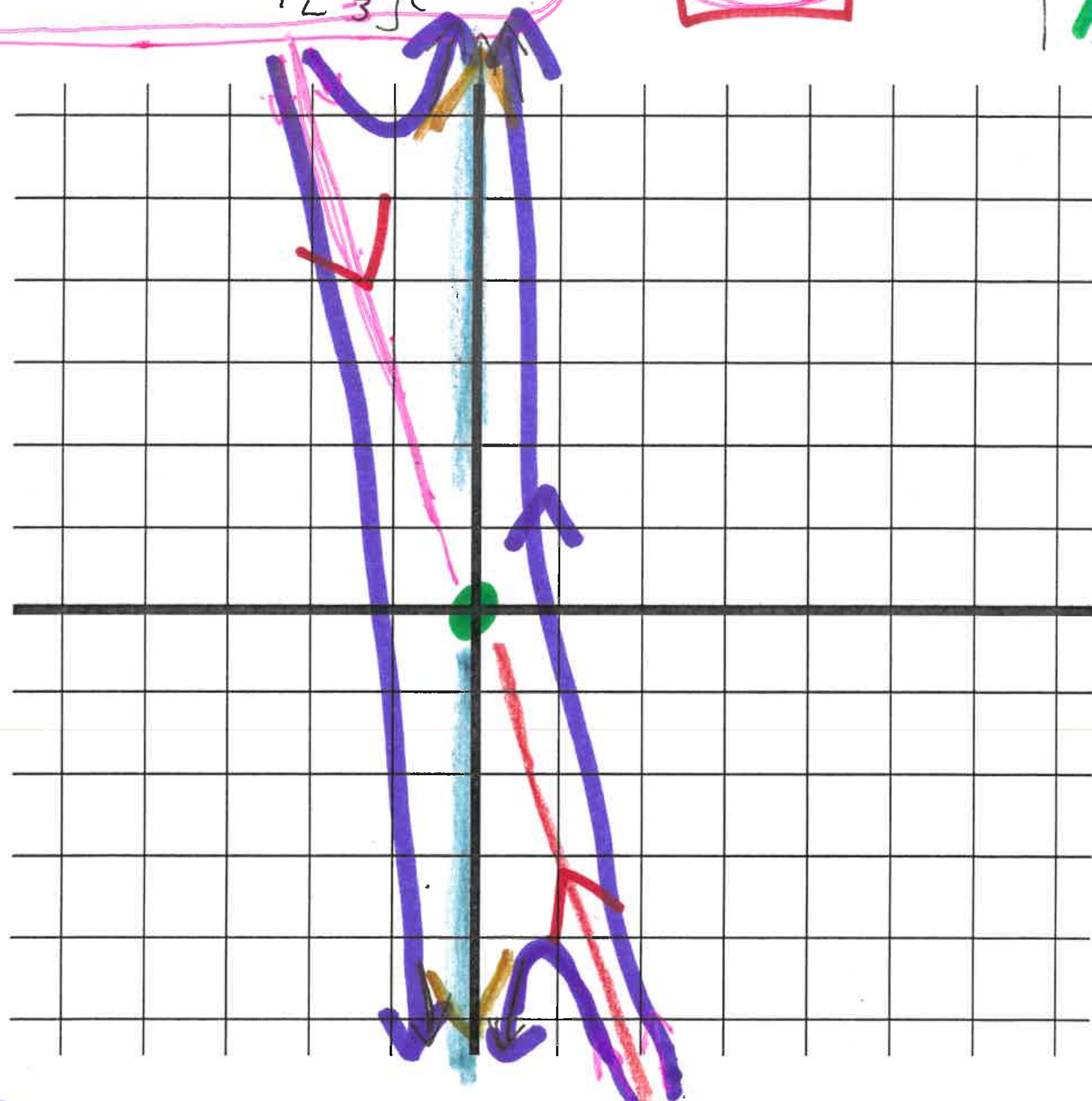
Example 1: Given that the solution to to $\mathbf{x}' = \begin{bmatrix} -2 & 0 \\ 21 & 5 \end{bmatrix} \mathbf{x}$ is $\mathbf{x} = c_1 \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{5t}$

$\boxed{\text{As } t \to +\infty \quad \vec{x} \to c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{5t}}$

As $t \to -\infty \quad \vec{x} \to c_1 \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{-2t}$

$3/-1$

$1/0$



9.1

The equilibrium solution for this system of equations is $\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

This equilibrium solution is (choose one): asymptotically stable or unstable

go away from equilibrium soln $\vec{x} = 0$ for most trajectories

Classify this critical point's type: SADDLE one positive and one negative e. value

$c_1 \neq 0$
$c_2 \neq 0$

Example 1: Given that the solution to to $\mathbf{x}' = \begin{bmatrix} -2 & 0 \\ 21 & 5 \end{bmatrix} \mathbf{x}$ is $\mathbf{x} = c_1 \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{5t}$

At $t \to +\infty$, $\cancel{c_1 \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{-2t}} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{5t}$ approaches $c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{5t}$

$\to 0$

going forwards in time

$t = 10$, e^{-20} tiny

e^{50} large

At $t \to -\infty$, $c_1 \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{-2t} + \cancel{c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{5t}}$ approaches $c_1 \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{-2t}$

$\to 0$

going backwards in time

$t = -10$ $\quad e^{-2(-10)} = e^{20}$ large

$e^{5(-10)} = e^{-50}$ tiny

Ch 1 and section 2.5 constant soln

Find the **equilibrium solution(s)** for $\mathbf{x}' = A\mathbf{x}$ (Recall equilibrium solns are constant solns)

Recall a solution is an equilibrium solution iff $\mathbf{x}(t) = \mathbf{C}$ iff $\mathbf{x}'(t) = 0$

constant

Setting $\mathbf{x}' = 0$, implies $\mathbf{0} = A\mathbf{x}$.

Thus $\mathbf{x} = \mathbf{C}$ is an equilibrium solution iff it is a solution to $\mathbf{0} = A\mathbf{x}$.

$\mathbf{x}' = 0$

line w/slope 0 ie a horizontal line in 3d which projects to a single point in the $x_1 x_2$ plane

Case 1 (not emphasized/covered): $det(A) = 0$.

In this case, $A\mathbf{x} = \mathbf{0}$ has an infinite number of solutions. Note this case corresponds to the case when 0 is an eigenvalue of A since there are nonzero solutions to $A\mathbf{v} = 0\mathbf{v}$

Case 2: $det(A) \neq 0$. $\Rightarrow$ unique soln to $Ax = 0$

Then $A\mathbf{x} = \mathbf{0}$ has a unique solution, $\mathbf{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Thus if $det(A) \neq 0$, $\mathbf{x} = \vec{0}$ is the only equilibrium solution of $\mathbf{x}' = A\mathbf{x}$

$\vec{x}(t) = \vec{0}$ is a soln to $\vec{x}' = A\vec{x}$

Slope fields:

* For complex eigenvalue case, one slope is needed.

* For real eigenvalue case, 0 and ∞ slopes can be helpful and can catch graphing errors, but your graph does **not** need to be that accurate.

For $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 21 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} =$

$\dfrac{dx_1}{dt} =$

$\dfrac{dx_2}{dt} =$

$\dfrac{dx_2}{dx_1} =$

Slope 0:

Slope ∞:

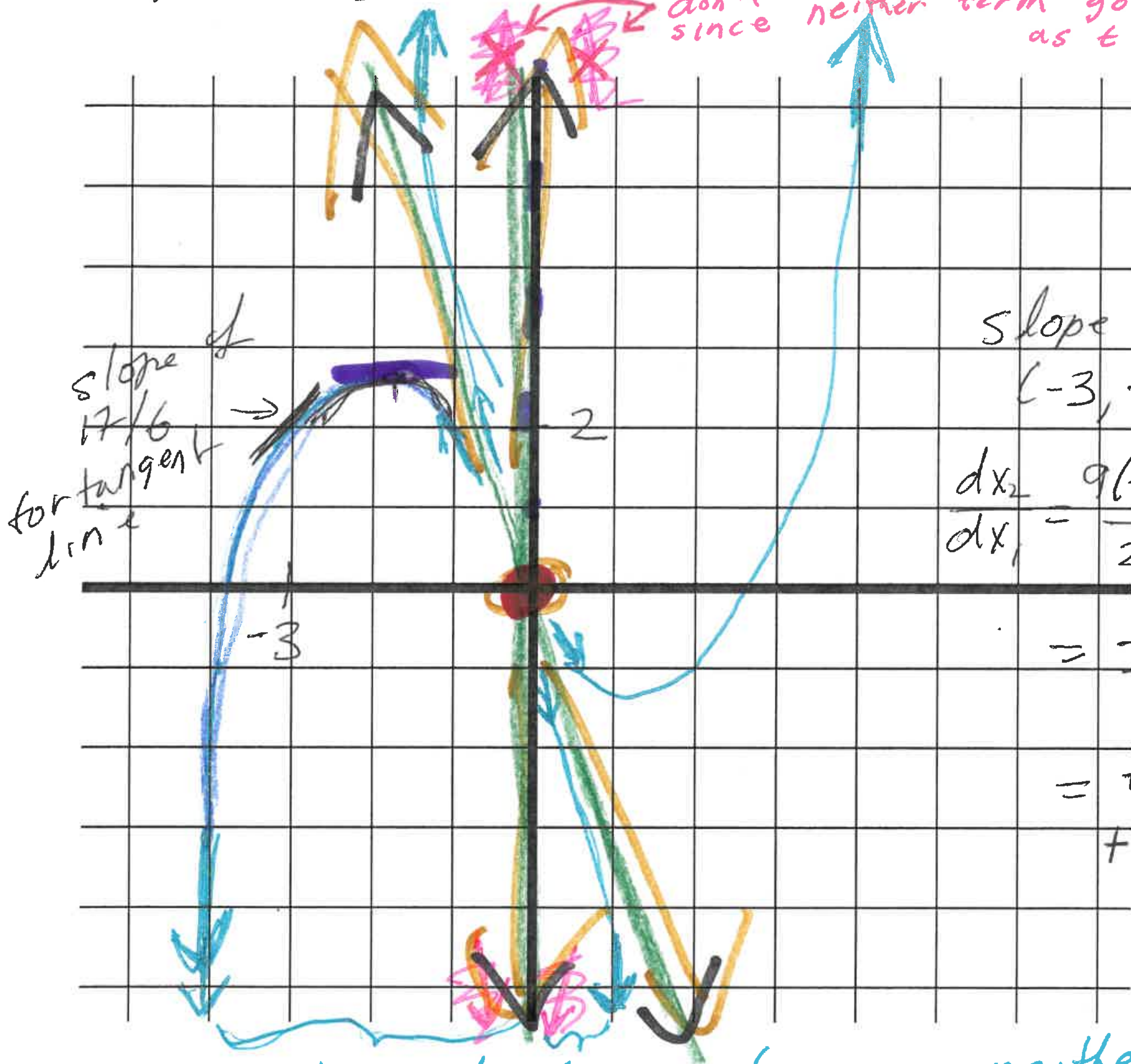
7.5 Two real eigenvalues (Example 2: Two **positive** eigenvalues).

Example 3: Given that the solution to $\mathbf{x}' = \begin{bmatrix}2 & 0\\ 9 & 5\end{bmatrix}\mathbf{x}$ is $\mathbf{x} = c_1\begin{bmatrix}0\\1\end{bmatrix}e^{5t} + c_2\begin{bmatrix}-1\\3\end{bmatrix}e^{2t}$

As $t \to +\infty$, $\vec{x} \to c_1\begin{bmatrix}0\\1\end{bmatrix}e^{5t} + \vec{d}$ ← displacement

As $t \to -\infty$, $\vec{x} \to c_2\begin{bmatrix}-1\\3\end{bmatrix}e^{2t}$

don't want small displacement since neither term goes to 0 as $t \to +\infty$



slope at $(-3, 2)$

$$\frac{dx_2}{dx_1} = \frac{9(-3)+5(2)}{2(-3)} = \frac{-27+10}{-6} = \frac{+17}{+6}$$

large displacement since neither term goes to $\vec{0}$ as $t \to +\infty$

The equilibrium solution for this system of equations is $\begin{bmatrix}x_1(t)\\x_2(t)\end{bmatrix} =$

This equilibrium solution is (choose one): asymptotically stable or unstable

Classify this critical point's type:

Two + e. values

$\boxed{\text{ASSUME } c_1 \neq 0 \text{ and } c_2 \neq 0}$

Example 3: Given that the solution to $\mathbf{x}' = \begin{bmatrix} 2 & 0 \\ 9 & 5 \end{bmatrix} \mathbf{x}$ is $\mathbf{x} = c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{5t} + c_2 \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{2t}$

At $t \to +\infty$, $c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{5t} + c_2 \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{2t}$ approaches $c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{5t} + \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$

$t = 10 \quad e^{5t} = e^{50}, \; e^{2t} = e^{20}$

very very large $\to e^{50} >> e^{20} \leftarrow$ large

At $t \to -\infty$, $c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{5t} + c_2 \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{2t}$ approaches $c_2 \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{2t}$

very tiny

small

$t = 10 \quad e^{-50} << e^{-20}$

both go to zero, but $e^{5t} \to 0$ faster than e^{2t}

$\boxed{\text{ASSUME } c_1 \neq 0 \text{ and } c_2 \neq 0}$

Two Positive e. values

Example 3: Given that the solution to $\mathbf{x}' = \begin{bmatrix} 2 & 0 \\ 9 & 5 \end{bmatrix} \mathbf{x}$ is $\mathbf{x} = c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{5t} + c_2 \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{2t}$

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dominates

displacement for $c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{5t}$

At $t \to +\infty$, $c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{5t} + c_2 \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{2t}$ approaches $c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{5t} + \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$

forwards in time

very very large

large

displacement

$t = 10$: $e^{50} \gg e^{20}$

very very large

large

as $t \to +\infty$ $e^{5t} \gg e^{2t}$

displacement

At $t \to -\infty$, $c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{5t} + c_2 \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{2t}$ approaches $c_2 \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{2t}$

backwards in time

very very tiny

small

$t = -10$ $e^{-50} \ll e^{-20}$

very tiny

small

For $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ -9 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} =$

$\frac{dx_1}{dt} =$

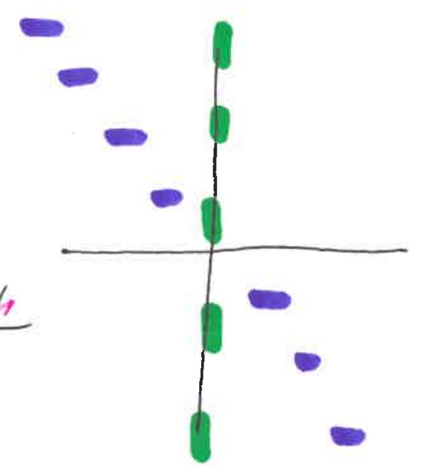
$\frac{dx_2}{dt} =$

$\frac{dx_2}{dx_1} =$

Slope 0:

Slope ∞:

For $\boxed{\begin{bmatrix} x_1' \\ x_2' \end{bmatrix}} = \begin{bmatrix} 2 & 0 \\ 9 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ \boxed{9x_1 + 5x_2} \end{bmatrix}$



$\frac{dx_1}{dt} = 2x_1$

$\frac{dx_2}{dt} = 9x_1 + 5x_2$

$\frac{dx_2/\cancel{dt}}{dx_1/\cancel{dt}}$

$\frac{dx_2}{dx_1} = \frac{9x_1 + 5x_2}{2x_1}$

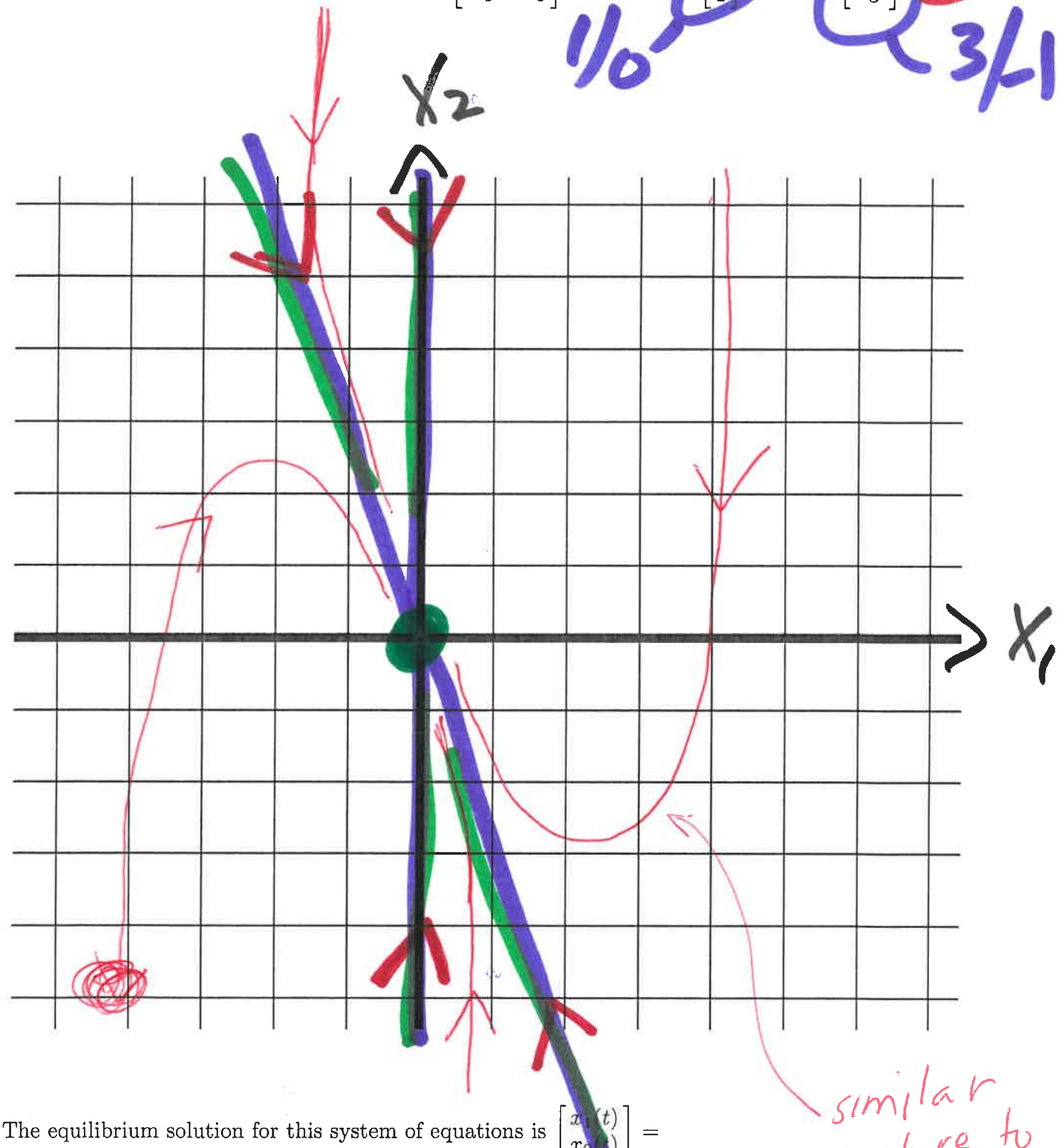
Slope 0: $9x_1 + 5x_2 = 0 \Rightarrow x_2 = \frac{-9x_1}{5}$

Slope ∞: $\boxed{x_1 = 0}$

7.5 Two real eigenvalues (Example 3: Two **negative** eigenvalues).

Example 2: Given that the solution to $\mathbf{x}' = \begin{bmatrix} -2 & 0 \\ -9 & -5 \end{bmatrix} \mathbf{x}$ is $\mathbf{x} = c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-5t} + c_2 \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{-2t}$

1/0 3/-1



The equilibrium solution for this system of equations is $\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} =$

similar picture to positive e. value case but w/ arrows reversed

This equilibrium solution is (choose one): asymptotically stable or unstable

Classify this critical point's type:

$$\vec{X}' = \begin{bmatrix} 3 & -13 \\ 5 & 1 \end{bmatrix} \vec{X}$$

To solve

① Find e.values ② Find e.vectors

To graph

① Find e.value ② use 1 slope for direction

clockwise or counter clockwise

Find e.values $\det(A - rI) = 0$

$$\begin{vmatrix} 3-r & -13 \\ 5 & 1-r \end{vmatrix} = (3-r)(1-r) - 5(-13)$$

$$= r^2 - 4r + 3 + 65$$

$$= r^2 - 4r + 68 = 0$$

$$r = \frac{4 \pm \sqrt{16 - 4(1)(68)}}{2(1)} = \underbrace{2}_{e^{2t}} \pm 8i$$

spiral out

Graph will be either