$$g(t) = \begin{cases} 0 & t < 4 \\ 2 & 4 \le t < 10 \\ t & t \ge 10 \end{cases}$$

Hence $g(t) = 2u_4(t) + (t-2)u_{10}(t)$

$$\text{Solve } \boxed{3y'' + y' + y = 2u_4(t) + (t-2)u_{10}(t),} \quad y(0) = 0,\ y'(0) = 0.$$

$$3\mathcal{L}(y'') + \mathcal{L}(y') + \mathcal{L}(y) = \mathcal{L}(2u_4(t)) + \mathcal{L}((t-2)u_{10}(t))$$

$$\text{Thm: } \mathcal{L}(u_c(t)f(t-c)) = e^{-cs}\mathcal{L}(f(t)).$$

+c

Thus $\mathcal{L}(u_c(t)f(t)) =$ $\mathcal{L}(u_c(t)f(t+c))$

$$3[s^2\mathcal{L}(y) - sy(0) - y'(0)] + s\mathcal{L}(y) - y(0) + \mathcal{L}(y) = e^{-4s}\mathcal{L}(2) + e^{-10s}\mathcal{L}((t+8))$$

$$3[s^2\mathcal{L}(y)] + s\mathcal{L}(y) + \mathcal{L}(y) = 2e^{-4s}\mathcal{L}(1) + e^{-10s}\mathcal{L}(t) + 8e^{-10s}\mathcal{L}(1)$$

$$\mathcal{L}(y)[3s^2 + s + 1] = e^{-4s}\frac{2}{s} + e^{-10s}\frac{1}{s^2} + e^{-10s}\frac{8}{s}$$

← characteristic polynomial

$$\mathcal{L}(y) = e^{-4s}\frac{2}{s[3s^2+s+1]} + e^{-10s}\frac{1}{s^2[3s^2+s+1]} + 8e^{-10s}\frac{1}{s[3s^2+s+1]}$$

$$y = 2\mathcal{L}^{-1}\left(e^{-4s}\frac{1}{s[3s^2+s+1]}\right) + \mathcal{L}^{-1}\left(e^{-10s}\frac{1}{s^2[3s^2+s+1]}\right) + 8\mathcal{L}^{-1}\left(e^{-10s}\frac{1}{s[3s^2+s+1]}\right)$$

$e^{-cs}F(s)$ $\quad$ $e^{-10s}H(s)$ $\quad$ $e^{-cs}F(s)$

$c = 4$ $\quad$ $c = 10$

$= 2u_4 f(t-4) + u_{10} h(t-10) + 8u_{10} f(t-10)$

$$y = 2u_4(t)f(t-4) + u_{10}h(t-10) + 8u_{10}f(t-10)$$

where $f(t) = \mathcal{L}^{-1}\left(\frac{1}{s[3s^2+s+1]}\right)$ and $h(t) = \mathcal{L}^{-1}\left(\frac{1}{s^2[3s^2+s+1]}\right)$

factors   factors

$$\frac{1}{s[3s^2+s+1]} = \frac{A}{s} + \frac{Bs+C}{3s^2+s+2}$$

⇐ Does not factor further

partial fractions

$$1 = A(3s^2+s+1) + (Bs+C)s$$

$$0s^2 + 0s + 1 = (3A+B)s^2 + (A+C)s + A$$

$$0 = 3A+B,\ 0 = A+C,\ 1 = A$$

Hence $A = 1$, $B = -3A = -3$, $C = -A = -1$

$$f(t) = \mathcal{L}^{-1}\left(\frac{1}{s[3s^2+s+1]}\right)$$

partial fractions

$$= \mathcal{L}^{-1}\left(\frac{1}{s} + \frac{-3s-1}{3s^2+s+1}\right)$$

$$= \mathcal{L}^{-1}\left(\frac{1}{s}\right) + \mathcal{L}^{-1}\left(\frac{-3s-1}{3s^2+s+1}\right)$$

$$= 1 + \mathcal{L}^{-1}\left(\frac{-3s-1}{3[s^2+\frac{1}{3}s+\frac{1}{3}]}\right)$$

$\div 2$

$$= 1 + \mathcal{L}^{-1}\left(\frac{-3s-1}{3[(s^2+\frac{1}{3}s+\frac{1}{36}) - \frac{1}{36}+\frac{1}{3}]}\right)$$

$$= 1 + \mathcal{L}^{-1}\left(\frac{-3s-1}{3[(s+\frac{1}{6})^2-\frac{1}{36}+\frac{1}{3}]}\right)$$

$$= 1 + \mathcal{L}^{-1}\left(\frac{-3(s+\frac{1}{3})}{3[(s+\frac{1}{6})^2+\frac{11}{36}]}\right)$$

$$= 1 + \mathcal{L}^{-1}\left(\frac{-(s+\frac{1}{6}-\frac{1}{6}+\frac{1}{3})}{[(s+\frac{1}{6})^2+\frac{11}{36}]}\right)$$

$$\mathcal{L}^{-1}\left(\frac{s-a}{(s-a)^2+b^2}\right) \Rightarrow e^{at}\cos bt$$

$$\mathcal{L}^{-1}\left(\frac{b}{(s-a)^2+b^2}\right) = e^{at}\sin bt$$

complete the square

$$3\left[s^2+\frac{2}{6}s+\frac{1}{36}+\frac{11}{36}\right] = 3s^2+s+1$$

$$= 1 + \mathcal{L}^{-1}\left(\frac{-(s+\frac{1}{6})+\frac{1}{6}}{[(s+\frac{1}{6})^2+\frac{11}{36}]}\right)$$

$$= 1 + \mathcal{L}^{-1}\left(\frac{-(s+\frac{1}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]} + \frac{+\frac{1}{6}}{[(s+\frac{1}{6})^2+\frac{11}{36}]}\right)$$

$b^2 = \frac{11}{36} \Rightarrow b = \frac{\sqrt{11}}{6}$

$$= 1 + \mathcal{L}^{-1}\left(\frac{-(s+\frac{1}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]} + \frac{+\frac{1}{6}\frac{6}{\sqrt{11}}\frac{\sqrt{11}}{6}}{[(s+\frac{1}{6})^2+\frac{11}{36}]}\right)$$

$$= 1 + \mathcal{L}^{-1}\left(\frac{-(s+\frac{1}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]} + \frac{+\frac{1}{\sqrt{11}}\frac{\sqrt{11}}{6}}{[(s+\frac{1}{6})^2+\frac{11}{36}]}\right)$$

$a = -\frac{1}{6}$ $\quad s-(-\frac{1}{6}$

Thm: $\mathcal{L}^{-1}(F(s-c)) = e^{ct}\mathcal{L}^{-1}(F(s))$ optional

$$= 1 + e^{-\frac{1}{6}t}\mathcal{L}^{-1}\left(\frac{-s}{[s^2+\frac{11}{36}]}\right) - \frac{1}{\sqrt{11}}e^{-\frac{1}{6}t}\mathcal{L}^{-1}\left(\frac{\frac{\sqrt{11}}{6}}{[s^2+\frac{11}{36}]}\right)$$

$$= 1 - e^{-\frac{1}{6}t}cos\frac{\sqrt{11}}{6}t - \frac{1}{\sqrt{11}}e^{-\frac{1}{6}t}sin\frac{\sqrt{11}}{6}t \quad = f(t)$$

$h(t) = \mathcal{L}^{-1}\left(\frac{1}{s^2[3s^2+s+1]}\right)$

$\frac{1}{s^2[3s^2+s+1]} = \frac{As+D}{s^2} + \frac{Bs+C}{3s^2+s+2} = \frac{A}{s} + \frac{D}{s^2} + \frac{Bs+C}{3s^2+s+2}$

$1 = (As+D)(3s^2+s+1) + (Bs+C)s^2$

$0s^3 + 0s^2 + 0s + 1 = (3A+B)s^3 + (A+3D+C)s^2 + (A+D)s + D$

$0 = 3A+B$, $0 = A+3D+C$, $0 = A+D$, $1 = D$.

Hence $D = 1$, $A = -D = -1$, $C = -A - 3D = 1 - 3 = -2$, $B = -3A = 3$.

$$\frac{1}{s^2[3s^2+s+1]} = \frac{-s+1}{s^2} + \left(\frac{3s-2}{3s^2+s+1}\right) = \frac{-s}{s^2} + \frac{1}{s^2} + \frac{3(s-\frac{2}{3})}{3[(s+\frac{1}{6})^2+\frac{11}{36}]}$$

$$= \frac{-1}{s} + \frac{1}{s^2} + \frac{(s+\frac{1}{6}-\frac{1}{6}-\frac{2}{3})}{[(s+\frac{1}{6})^2+\frac{11}{36}]} = \frac{-1}{s} + \frac{1}{s^2} + \frac{(s+\frac{1}{6}-\frac{5}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]}$$

$$= \frac{-1}{s} + \frac{1}{s^2} + \frac{(s+\frac{1}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]} - \frac{(\frac{5}{6})(\frac{6}{\sqrt{11}}\frac{\sqrt{11}}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]}$$

$$= \frac{-1}{s} + \frac{1}{s^2} + \frac{(s+\frac{1}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]} - \frac{(\frac{5}{6})(\frac{6}{\sqrt{11}}\frac{\sqrt{11}}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]}$$

$$= \frac{-1}{s} + \frac{1}{s^2} + \frac{(s+\frac{1}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]} - \frac{\frac{5}{\sqrt{11}}(\frac{\sqrt{11}}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]}$$

$$h(t) = \mathcal{L}^{-1}\left(\frac{1}{s^2[3s^2+s+1]}\right)$$

$$= \mathcal{L}^{-1}\left(\frac{-1}{s} + \frac{1}{s^2} + \frac{(s+\frac{1}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]} - \frac{\frac{5}{\sqrt{11}}(\frac{\sqrt{11}}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]}\right)$$

$$= -1 + t + e^{-\frac{1}{6}t} cos\frac{\sqrt{11}}{6}t - \frac{5}{\sqrt{11}}e^{-\frac{1}{6}t} sin\frac{\sqrt{11}}{6}t \;= h(t)$$

Hence the final answer is

$$y = 2u_4(t)f(t-4) + u_{10}h(t-10) + 8u_{10}f(t-10)$$

$$= 2u_4(t)[1-e^{-\frac{1}{6}(t-4)}cos\frac{\sqrt{11}}{6}(t-4)-\frac{1}{\sqrt{11}}e^{-\frac{1}{6}(t-4)}sin\frac{\sqrt{11}}{6}(t-4)]+$$
$$u_{10}[-1+t-10+e^{-\frac{1}{6}(t-10)}cos\frac{\sqrt{11}}{6}(t-10)-\frac{5}{\sqrt{11}}e^{-\frac{1}{6}(t-10)}sin\frac{\sqrt{11}}{6}(t-10)]$$
$$+8u_{10}[1-e^{-\frac{1}{6}(t-10)}cos\frac{\sqrt{11}}{6}(t-10)-\frac{1}{\sqrt{11}}e^{-\frac{1}{6}(t-10)}sin\frac{\sqrt{11}}{6}(t-10)]$$

Simp

$$g(t) = \begin{cases} 0 & t < 4 \\ 2 & 4 \le t < 10 \\ t & t \ge 10 \end{cases}$$

Hence $g(t) = 2u_4(t) + (t-2)u_{10}(t)$

$$\text{Solve } \mathcal{L}(3y'' + y' + y) = \mathcal{L}(2u_4(t) + (t-2)u_{10}(t)),$$
$$y(0) = 0,\ y'(0) = 0.$$

$$3\mathcal{L}(y'') + \mathcal{L}(y') + \mathcal{L}(y) = \mathcal{L}(2u_4(t)) + \mathcal{L}((t-2)u_{10}(t))$$

$+c$

$$\text{Thm: } \boxed{\mathcal{L}(u_c(t)f(t-c)) = e^{-cs}\mathcal{L}(f(t)). = e^{-cs}F(s)}$$

$$\text{Thus } \mathcal{L}(u_c(t)f(t)) = \underline{e^{-cs}\mathcal{L}(f(t+c))}.$$

$$3[s^2\mathcal{L}(y) - \cancel{sy(0)}^{0} - \cancel{y'(0)}^{0}] + s\mathcal{L}(y) - \cancel{y(0)}^{0} + \mathcal{L}(y)$$
$$= e^{-4s}\mathcal{L}(2) + e^{-10s}\mathcal{L}((t+8)1)$$

$$3[s^2\underline{\mathcal{L}(y)}] + s\underline{\mathcal{L}(y)} + \underline{\mathcal{L}(y)} = 2e^{-4s}\mathcal{L}(1) + e^{-10s}\mathcal{L}(t) + 8e^{-10s}\mathcal{L}(1)$$

$$\mathcal{L}(y)\underbrace{[3s^2 + s + 1]}_{\text{characteristic poly}} = e^{-4s}\frac{2}{s} + e^{-10s}\frac{1}{s^2} + e^{-10s}\frac{8}{s}$$

$$\mathcal{L}^{-1}(\mathcal{L}(y)) = \mathcal{L}^{-1}\left(e^{-4s}\frac{2}{s[3s^2+s+1]} + e^{-10s}\frac{1}{s^2[3s^2+s+1]} + 8e^{-10s}\frac{1}{s[3s^2+s+1]}\right)$$

$$y = 2\mathcal{L}^{-1}\Big(\underbrace{e^{-4s}\frac{1}{s[3s^2+s+1]}}_{e^{-4s}F(s)}\Big) + \mathcal{L}^{-1}\Big(\underbrace{e^{-10s}\frac{1}{s^2[3s^2+s+1]}}_{e^{-10s}H(s)}\Big)$$
$$+8\mathcal{L}^{-1}\Big(\underbrace{e^{-10s}\frac{1}{s[3s^2+s+1]}}_{e^{-10s}F(s)}\Big)$$

$$= 2u_4(t)f(t-4) + u_{10}(t)h(t-10) + 8u_{10}(t)f(t-10)$$

$c = 4$ $\qquad$ $c = 10$ $\qquad$ $c = 10$

$$y = 2u_4(t)f(t-4) + u_{10}h(t-10) + 8u_{10}f(t-10)$$

where $f(t) = \mathcal{L}^{-1}\left(\frac{1}{s[3s^2+s+1]}\right)$ and $h(t) = \mathcal{L}^{-1}\left(\frac{1}{s^2[3s^2+s+1]}\right)$

factors over $\mathbb{R}$

$$\frac{1}{s[3s^2+s+1]} = \frac{A}{s} + \frac{Bs+C}{3s^2+s+2}$$

partial fractions

$$1 = A(3s^2+s+1) + (Bs+C)s$$

$$0s^2 + 0s + 1 = (3A+B)s^2 + (A+C)s + \underline{A}$$

$$0 = 3A + B,\ 0 = A + C,\ 1 = A$$

Hence $\underline{A = 1}$, $B = -3A = \underline{-3}$, $C = -A = \underline{-1}$

$$f(t) = \mathcal{L}^{-1}\left(\frac{1}{s[3s^2+s+1]}\right)$$

partial fractions

$$= \mathcal{L}^{-1}\left(\frac{1}{s} + \frac{-3s-1}{3s^2+s+1}\right)$$

$$\mathcal{L}^{-1}\left(\frac{s-a}{(s-a)^2+b^2}\right) = e^{at}\cos bt$$

$$= \mathcal{L}^{-1}\left(\frac{1}{s} + \frac{-3s-1}{3s^2+s+1}\right)$$

$$\mathcal{L}^{-1}\left(\frac{b}{(s-a)^2+b^2}\right) = e^{at}\sin bt$$

$$= 1 + \mathcal{L}^{-1}\left(\frac{-3s-1}{3[s^2+\frac{1}{3}s+\frac{1}{3}]}\right)$$

$$= 1 + \mathcal{L}^{-1}\left(\frac{-3s-1}{3[(s^2+\frac{1}{3}s+\underline{1/36})-\underline{1/36}+\frac{1}{3}]}\right)$$

$\div 2$

$$= 1 + \mathcal{L}^{-1}\left(\frac{-3s-1}{3[(s+\frac{1}{6})^2-\frac{1}{36}+\frac{1}{3}]}\right)$$

$$= 1 + \mathcal{L}^{-1}\left(\frac{-\cancel{3}(s+\frac{1}{3})}{\cancel{3}[(s+\frac{1}{6})^2+\frac{11}{36}]}\right)$$

Check

$$3\left[(s+\tfrac{1}{6})^2 + \tfrac{11}{36}\right]$$

$$3\left[s^2 + \tfrac{1}{3}s + \tfrac{1}{36} + \tfrac{11}{36}\right]$$

$$= 1 + \mathcal{L}^{-1}\left(\frac{-(s+\frac{1}{6}-\frac{1}{6}+\frac{1}{3})}{[(s+\frac{1}{6})^2+\frac{11}{36}]}\right)$$

$$= 3s^2 + s + \frac{12}{12} = 3s^2+s+1\ ✓$$

check: $s+\frac{1}{3}$

$$\frac{(s-a)}{(s-a)^2+b^2} = \frac{s+\frac{1}{6}}{(s+\frac{1}{6})^2+\frac{11}{36}}$$

$$= 1 - \mathcal{L}^{-1}\left(\frac{-(s+\frac{1}{6}+\frac{1}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]}\right)$$

$$\frac{b}{(s-a)^2+b^2} = \frac{\sqrt{11}/6}{(s+\frac{1}{6})^2+\frac{11}{36}}$$

$$= 1 - \mathcal{L}^{-1}\left(\frac{+(s+\frac{1}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]} + \frac{+\frac{1}{6}}{[(s+\frac{1}{6})^2+\frac{11}{36}]}\right)$$

$b^2 = \frac{11}{36} \Rightarrow b = \sqrt{\ }$

$$= 1 - \mathcal{L}^{-1}\left(\frac{+(s+\frac{1}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]} + \frac{+\frac{1}{6}\frac{6}{\sqrt{11}}(\frac{\sqrt{11}}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]}\right)$$

$$= 1 - \mathcal{L}^{-1}\left(\frac{+(s+\frac{1}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]} + \frac{+\frac{1}{\sqrt{11}}(\frac{\sqrt{11}}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]}\right)$$

Thm: $\mathcal{L}^{-1}(F(s-c)) = e^{ct}\mathcal{L}^{-1}(F(s))$ optional

$$= 1 - e^{-\frac{1}{6}t}\mathcal{L}^{-1}\left(\frac{+s}{[s^2+\frac{11}{36}]}\right) - \frac{1}{\sqrt{11}}e^{-\frac{1}{6}t}\mathcal{L}^{-1}\left(\frac{\frac{\sqrt{11}}{6}}{[s^2+\frac{11}{36}]}\right)$$

$$= 1 - e^{-\frac{1}{6}t}cos\frac{\sqrt{11}}{6}t - \frac{1}{\sqrt{11}}e^{-\frac{1}{6}t}sin\frac{\sqrt{11}}{6}t = f(t)$$

$a = -\frac{1}{6}$, $b = \frac{\sqrt{11}}{6}$

$h(t) = \mathcal{L}^{-1}\left(\frac{1}{s^2[3s^2+s+1]}\right)$ factors $\Rightarrow$ partial fracti

$$\frac{1}{s^2[3s^2+s+1]} = \frac{As+D}{s^2} + \frac{Bs+C}{3s^2+s+2} = \frac{A}{s} + \frac{D}{s^2} + \frac{Bs+C}{3s^2+s+2}$$

$1 = (As+D)(3s^2+s+1) + (Bs+C)s^2$

$0s^3 + 0s^2 + 0s + 1 = (3A+B)s^3 + (A+3D+C)s^2 + (A+D)s + D$

$0 = 3A+B$, $0 = A+3D+C$, $0 = A+D$, $1 = D$.

Hence $D = 1$, $A = -D = -1$, $C = -A - 3D = 1 - 3 = -2$, $B = -3A = 3$.

$$\frac{1}{s^2[3s^2+s+1]} = \frac{-s+1}{s^2} + \frac{3s-2}{3s^2+s+1} = \frac{-s}{s^2} + \frac{1}{s^2} + \frac{3(s-\frac{2}{3})}{3[(s+\frac{1}{6})^2+\frac{11}{36}]}$$

$$= \frac{-1}{s} + \frac{1}{s^2} + \frac{(s+\frac{1}{6}-\frac{1}{6}-\frac{2}{3})}{[(s+\frac{1}{6})^2+\frac{11}{36}]} = \frac{-1}{s} + \frac{1}{s^2} + \frac{(s+\frac{1}{6}-\frac{5}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]}$$

$$= \frac{-1}{s} + \frac{1}{s^2} + \frac{(s+\frac{1}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]} - \frac{(\frac{5}{6})(\frac{6}{\sqrt{11}}\frac{\sqrt{11}}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]}$$

$$= \frac{-1}{s} + \frac{1}{s^2} + \frac{(s+\frac{1}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]} - \frac{(\frac{5}{6})(\frac{6}{\sqrt{11}}\frac{\sqrt{11}}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]}$$

$$= \frac{-1}{s} + \frac{1}{s^2} + \frac{(s+\frac{1}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]} - \frac{\frac{5}{\sqrt{11}}(\frac{\sqrt{11}}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]}$$

$$h(t) = \mathcal{L}^{-1}\left(\frac{1}{s^2[3s^2+s+1]}\right)$$

$$= \mathcal{L}^{-1}\left(\frac{-1}{s} + \frac{1}{s^2} + \frac{(s+\frac{1}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]} - \frac{\frac{5}{\sqrt{11}}(\frac{\sqrt{11}}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]}\right)$$

$$= -1 + t + e^{-\frac{1}{6}t} cos\frac{\sqrt{11}}{6}t - \frac{5}{\sqrt{11}} e^{-\frac{1}{6}t} sin\frac{\sqrt{11}}{6}t \quad = h(t)$$

Hence the final answer is

$$y = 2u_4(t)f(t-4) + u_{10}h(t-10) + 8u_{10}f(t-10)$$

$$= 2u_4(t)[1-e^{-\frac{1}{6}(t-4)} cos\frac{\sqrt{11}}{6}(t-4) - \frac{1}{\sqrt{11}} e^{-\frac{1}{6}(t-4)} sin\frac{\sqrt{11}}{6}(t-4)] +$$
$$u_{10}[-1+t-10+e^{-\frac{1}{6}(t-10)} cos\frac{\sqrt{11}}{6}(t-10) - \frac{5}{\sqrt{11}} e^{-\frac{1}{6}(t-10)} sin\frac{\sqrt{11}}{6}(t-10)]$$
$$+8u_{10}[1-e^{-\frac{1}{6}(t-10)} cos\frac{\sqrt{11}}{6}(t-10) - \frac{1}{\sqrt{11}} e^{-\frac{1}{6}(t-10)} sin\frac{\sqrt{11}}{6}(t-10)]$$

$$g(t) = \begin{cases} 0 & t < 4 \\ 2 & 4 \le t < 10 \\ t & t \ge 10 \end{cases}$$

Hence $g(t) = 2u_4(t) + (t-2)u_{10}(t)$

$$\boxed{\text{Solve } 3y'' + y' + y = 2u_4(t) + (t-2)u_{10}(t), \quad y(0) = 0,\ y'(0) = 0.}$$

$$3\mathcal{L}(y'') + \mathcal{L}(y') + \mathcal{L}(y) = \mathcal{L}(2u_4(t)) + \mathcal{L}((t-2)u_{10}(t))$$

$$\boxed{\text{Thm: } \mathcal{L}(u_c(t)f(t-c)) = e^{-cs}\mathcal{L}(f(t)).} = e^{-cs}F(s)$$

Thus $\mathcal{L}(u_c(t)f(t)) = e^{-cs}\mathcal{L}(f(t+c))$.

$$3[s^2\mathcal{L}(y) - sy(0) - y'(0)] + s\mathcal{L}(y) - y(0) + \mathcal{L}(y) = e^{-4s}\mathcal{L}(2) + e^{-10s}\mathcal{L}((t+8))$$

$$3[s^2\mathcal{L}(y)] + s\mathcal{L}(y) + \mathcal{L}(y) = 2e^{-4s}\mathcal{L}(1) + e^{-10s}\mathcal{L}(t) + 8e^{-10s}\mathcal{L}(1)$$

$$\mathcal{L}(y)\underbrace{[3s^2 + s + 1]}_{\text{characteristic poly}} = e^{-4s}\frac{2}{s} + e^{-10s}\frac{1}{s^2} + e^{-10s}\frac{8}{s}$$

$$\mathcal{L}^{-1}\ \mathcal{L}(y) = \mathcal{L}^{-1}\ e^{-4s}\frac{2}{s[3s^2+s+1]} + e^{-10s}\frac{1}{s^2[3s^2+s+1]} + 8e^{-10s}\frac{1}{s[3s^2+s+1]}$$

$$y = 2\mathcal{L}^{-1}\left(e^{-4s}\frac{1}{s[3s^2+s+1]}\right) + \mathcal{L}^{-1}\left(e^{-10s}\frac{1}{s^2[3s^2+s+1]}\right) + 8\mathcal{L}^{-1}\left(e^{-10s}\frac{1}{s[3s^2+s+1]}\right)$$

$$= 2\mathcal{L}^{-1}(e^{-4s}F(s)) + \mathcal{L}^{-1}(e^{-10s}H(s)) + 8\mathcal{L}^{-1}(e^{-10s}F(s))$$

$$= 2u_4 f(t-4) + u_{10}h(t-10) + 8u_{10}f(t-10)$$

$f(t-10)$

$$\boxed{y = 2u_4(t)f(t-4) + u_{10}h(t-10) + 8u_{10}f(t-10)}$$

where $f(t) = \mathcal{L}^{-1}(\frac{1}{s[3s^2+s+1]})$ and $h(t) = \mathcal{L}^{-1}(\frac{1}{s^2[3s^2+s+1]})$

factor $\Rightarrow$ partial fraction

$$\frac{1}{s[3s^2+s+1]} = \frac{A}{s} + \frac{Bs+C}{3s^2+s+2}$$

$$1 = A(3s^2+s+1) + (Bs+C)s$$

$$0s^2 + 0s + 1 = (3A+B)s^2 + (A+C)s + A$$

$$0 = 3A+B,\ 0 = A+C,\ 1 = A$$

Hence $A = 1$, $B = -3A = -3$, $C = -A = -1$

$$f(t) = \mathcal{L}^{-1}(\frac{1}{s[3s^2+s+1]})$$

partial fractions

$$= \mathcal{L}^{-1}(\frac{1}{s} + \frac{-3s-1}{3s^2+s+1})$$

Does not factor over $\mathbb{R}$ $\Rightarrow$ complete the square and use 9 & 10 and/or

$$\mathcal{L}^{-1}\left(\frac{s-a}{(s-a)^2+b^2}\right) = e^{at}\cos bt$$

$$\mathcal{L}^{-1}\left(\frac{b}{(s-a)^2+b^2}\right) = e^{at}\sin bt$$

$$= \mathcal{L}^{-1}(\frac{1}{s} + \frac{-3s-1}{3s^2+s+1})$$

$$= 1 + \mathcal{L}^{-1}(\frac{-3s-1}{3[s^2+\frac{1}{3}s+\frac{1}{3}]})$$

$$= 1 + \mathcal{L}^{-1}(\frac{-3s-1}{3[(s^2+\frac{1}{3}s+1/36)-1/36+\frac{1}{3}]})$$

$\div 2$

$$= 1 + \mathcal{L}^{-1}(\frac{-3s-1}{3[(s+\frac{1}{6})^2-\frac{1}{36}+\frac{1}{3}]})$$

$$= 1 \overset{-}{+} \mathcal{L}^{-1}(\frac{-\cancel{3}(s+\frac{1}{3})}{\cancel{3}[(s+\frac{1}{6})^2+\frac{11}{36}]})$$

check

$$3\left[(s+\tfrac{1}{6})^2 + \tfrac{11}{36}\right]$$

$$3\left[s^2 + \tfrac{2}{6}s + \tfrac{1}{36} + \tfrac{11}{36}\right]$$

$$= 3\left[s^2 + \tfrac{1}{3}s + \tfrac{12}{36}\right]$$

$$= 3s^2+s+1$$

$$= 1 \overset{-}{+} \mathcal{L}^{-1}(\frac{-(s+\frac{1}{6}-\frac{1}{6}+\frac{1}{3})}{[(s+\frac{1}{6})^2+\frac{11}{36}]})$$

$a = -\frac{1}{6}$

$b^2 = \frac{11}{36} \Rightarrow b = \frac{\sqrt{11}}{6}$

$$= 1 + \mathcal{L}^{-1}\left(\frac{+(s+\frac{1}{6})+\frac{1}{6}}{[(s+\frac{1}{6})^2+\frac{11}{36}]}\right)$$

$$= 1 + \mathcal{L}^{-1}\left(\frac{-(s+\frac{1}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]} + \frac{-\frac{1}{6}}{[(s+\frac{1}{6})^2+\frac{11}{36}]}\right)$$

$$= 1 + \mathcal{L}^{-1}\left(\frac{-(s+\frac{1}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]} + \frac{-\frac{1}{6}\left(\frac{6}{\sqrt{11}}\right)\left(\frac{\sqrt{11}}{6}\right)}{[(s+\frac{1}{6})^2+\frac{11}{36}]}\right)$$

$$= 1 + \mathcal{L}^{-1}\left(\frac{-(s+\frac{1}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]} + \frac{-\frac{1}{\sqrt{11}}\frac{\sqrt{11}}{6}}{[(s+\frac{1}{6})^2+\frac{11}{36}]}\right)$$

Thm: $\mathcal{L}^{-1}(F(s-c)) = e^{ct}\mathcal{L}^{-1}(F(s))$

$$= 1 + e^{-\frac{1}{6}t}\mathcal{L}^{-1}\left(\frac{-s}{[s^2+\frac{11}{36}]}\right) - \frac{1}{\sqrt{11}}e^{-\frac{1}{6}t}\mathcal{L}^{-1}\left(\frac{\frac{\sqrt{11}}{6}}{[s^2+\frac{11}{36}]}\right)$$

$$\boxed{= 1 - e^{-\frac{1}{6}t}cos\frac{\sqrt{11}}{6}t - \frac{1}{\sqrt{11}}e^{-\frac{1}{6}t}sin\frac{\sqrt{11}}{6}t = f(t)}$$

$h(t) = \mathcal{L}^{-1}\left(\frac{1}{s^2[3s^2+s+1]}\right)$

$\frac{1}{s^2[3s^2+s+1]} = \frac{As+D}{s^2} + \frac{Bs+C}{3s^2+s+2} = \frac{A}{s} + \frac{D}{s^2} + \frac{Bs+C}{3s^2+s+2}$

$1 = (As+D)(3s^2+s+1) + (Bs+C)s^2$

$0s^3 + 0s^2 + 0s + 1 = (3A+B)s^3 + (A+3D+C)s^2 + (A+D)s + D$

$0 = 3A+B,\ 0 = A+3D+C,\ 0 = A+D,\ 1 = D.$

Hence $D = 1$, $A = -D = -1$, $C = -A - 3D = 1 - 3 = -2$, $B = -3A = 3$.

$$\frac{1}{s^2[3s^2+s+1]} = \frac{-s+1}{s^2} + \left(\frac{3s-2}{3s^2+s+1}\right) = \frac{-\cancel{s}}{s^{\cancel{2}}} + \frac{1}{s^2} + \frac{3(s-\frac{2}{3})}{3[(s+\frac{1}{6})^2+\frac{11}{36}]}$$

$$= \frac{-1}{s} + \frac{1}{s^2} + \frac{(s+\frac{1}{6}-\frac{1}{6}-\frac{2}{3})}{[(s+\frac{1}{6})^2+\frac{11}{36}]} = \frac{-1}{s} + \frac{1}{s^2} + \frac{(s+\frac{1}{6}-\frac{5}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]}$$

$$= \frac{-1}{s} + \frac{1}{s^2} + \frac{(s+\frac{1}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]} - \frac{(\frac{5}{6})(\frac{6}{\sqrt{11}}\frac{\sqrt{11}}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]}$$

$$= \frac{-1}{s} + \frac{1}{s^2} + \frac{(s+\frac{1}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]} - \frac{(\frac{5}{6})(\frac{6}{\sqrt{11}}\frac{\sqrt{11}}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]}$$

$$= \frac{-1}{s} + \frac{1}{s^2} + \frac{(s+\frac{1}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]} - \frac{\frac{5}{\sqrt{11}}(\frac{\sqrt{11}}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]}$$

$$h(t) = \mathcal{L}^{-1}\left(\frac{1}{s^2[3s^2+s+1]}\right)$$

$$= \mathcal{L}^{-1}\left(\frac{-1}{s} + \frac{1}{s^2} + \frac{(s+\frac{1}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]} - \frac{\frac{5}{\sqrt{11}}(\frac{\sqrt{11}}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]}\right)$$

$$= -1 + t + e^{-\frac{1}{6}t} cos\frac{\sqrt{11}}{6}t - \frac{5}{\sqrt{11}} e^{-\frac{1}{6}t} sin\frac{\sqrt{11}}{6}t \quad = h(t)$$

Hence the final answer is

$$y = 2u_4(t)f(t-4) + u_{10}h(t-10) + 8u_{10}f(t-10)$$

$$= 2u_4(t)[1-e^{-\frac{1}{6}(t-4)}cos\frac{\sqrt{11}}{6}(t-4) - \frac{1}{\sqrt{11}}e^{-\frac{1}{6}(t-4)}sin\frac{\sqrt{11}}{6}(t-4)] +$$
$$u_{10}[-1+t-10+e^{-\frac{1}{6}(t-10)}cos\frac{\sqrt{11}}{6}(t-10) - \frac{5}{\sqrt{11}}e^{-\frac{1}{6}(t-10)}sin\frac{\sqrt{11}}{6}(t-10)] \blacksquare$$
$$+8u_{10}[1-e^{-\frac{1}{6}(t-10)}cos\frac{\sqrt{11}}{6}(t-10) - \frac{1}{\sqrt{11}}e^{-\frac{1}{6}(t-10)}sin\frac{\sqrt{11}}{6}(t-10)]$$