MATH:3600:0002 Final Exam
Dec. 12, 2016 SHOW ALL STEPS

[6] 1.) The Wronskian of $e^{-t}cos(t)$, $e^{-t}sin(t)$ is $W(e^{-t}cos(t), e^{-t}sin(t)) = \underline{\quad e^{-2t} \quad}$

$$W(e^{-t}cos(t), e^{-t}sin(t)) = \begin{vmatrix} e^{-t}cos(t) & e^{-t}sin(t) \\ -e^{-t}sin(t) - e^{-t}cos(t) & e^{-t}cos(t) - e^{-t}sin(t) \end{vmatrix}$$

$$= e^{-t}cos(t)[e^{-t}cos(t) - e^{-t}sin(t)] - e^{-t}sin(t)[-e^{-t}sin(t) - e^{-t}cos(t)]$$

$$= e^{-2t}cos^2(t) - e^{-2t}cos(t)sin(t) + e^{-2t}sin^2(t) + e^{-2t}cos(t)sin(t)$$

$$= e^{-2t}[cos^2(t) + sin^2(t)] = e^{-2t}$$

[14] 2.) Solve $t\frac{dy}{dt} + y = 8t^2$

$ty' + y = 8t^2$ is a first order linear DE.

Short method:

$[ty]' = ty' + y = 8t^2$

$[ty]' = 8t^2$

$\int [ty]' = \int 8t^2$

$ty = \frac{8}{3}t^3 + C$

Thus $y = \frac{8}{3}t^2 + Ct^{-1}$

Longer method:

$y' + \frac{1}{t}y = 8t$

Integrating factor: $u(t) = e^{\int \frac{dt}{t}} = e^{ln|t|} = |t|$

Multiply both sides by t: $ty' + y = 8t^2$ and continue as above.

Answer: $\underline{\quad y = \frac{8}{3}t^2 + Ct^{-1} \quad}$

[14] 3.) Solve $(3x^4 + 2y)dx + (2x + 4y^3)dy = 0$

$M_y = 2$ $N_x = 2$

If $\psi_x = 3x^4 + 2y$, then $\psi = \int(3x^4 + 2y)dx = \frac{3}{5}x^5 + 2xy + h(y)$.

$\psi_y = 2x + h'(y) = 2x + 4y^3$. Thus $h'(y) = 4y^3$ and $h(y) = y^4$

Answer: $\frac{3}{5}x^5 + 2xy + y^4 = C$

Check: $\frac{d}{dx}(\frac{3}{5}x^5 + 2xy + y^4) = \frac{d}{dx}(C)$

$3x^4 + 2y + 2x\frac{dy}{dx} + 4y^3\frac{dy}{dx} = 0$

Answer: ______________________________

[18] 4.) Solve $y''' - 4y' = 30e^{3t} + 8t$, $y(0) = 0$, $y'(0) = 6$, $y''(0) = 16$.

Solve homogeneous: $y''' - 4y' = 0$

$y = e^{rt}$: $r^3 - 4r = r(r^2 - 4) = r(r-2)(r+2) = 0$. Thus $r = 0, 2, -2$

General homogeneous solution: $y = c_1e^{0t} + c_2e^{2t} + c_3e^{-2t}$

Guess: $y = Ae^{3t}$: $y''' - 4y' = 27Ae^{3t} - 12e^{3t} = 15Ae3t = 30e^{3t}$. Thus $A = 2$.

Guess $y = At^2$. Thus $y' = 2At$ and $y''' = 0$. $0 - 4(2At) = 8t$. Thus $A = -1$

Answer: $y = y(t) = c_1e^{2t} + c_2e^{-2t} + c_3 - t^2 + 2e^{3t}$

$0 = c_1 + c_2 + c_3 + 2$

$6 = 2c_1 - 2c_2 + 6$

$16 = 4c_1 + 4c_2 - 2 + 18$

Answer: $y = y(t) = -2 - t^2 + 2e^{3t}$

Answer: ______________________________

[10] 5a.) Solve $\mathbf{x}' = \begin{pmatrix} 4 & 5 \\ 4 & 3 \end{pmatrix} \mathbf{x}$

Find eigenvalues:

$$\begin{vmatrix} 4-r & 5 \\ 4 & 3-r \end{vmatrix} = (4-r)(3-r) - 20 = r^2 - 7r + 12 - 20 = r^2 - 7r - 8 = (r-8)(r+1) = 0$$

$r = 8, -1$.

Find eigenvectors:

$r = 8$: $\begin{pmatrix} 4-8 & 5 \\ 4 & 3-8 \end{pmatrix}$

$$\begin{pmatrix} -4 & 5 \\ 4 & -5 \end{pmatrix} \begin{pmatrix} 5 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$r = -1$: $\begin{pmatrix} 4+1 & 5 \\ 4 & 3+1 \end{pmatrix}$

$$\begin{pmatrix} 5 & 5 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Answer: $y = c_1 e^{8t} \begin{pmatrix} 5 \\ 4 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

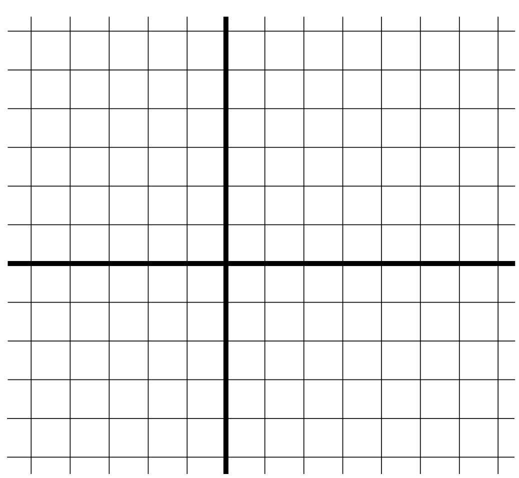
[4] 5b.) Find all equilibrium solutions and determine whether the critical point is asymptotically stable, stable, or unstable. Also classify it as to type (nodal source, nodal sink, saddle point, spiral source, spiral sink, center).

Equilibrium solution: $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Stability : *unstable*

Type: *saddle*

[4] 5c.) Plot a phase portrait for this system



[10] 6.) Given the recursive relation $a_n = \frac{4a_{n-1}}{(n+1)n}$ where $a_0 = 1$,

use induction to prove that $a_n = \frac{4^n}{(n+1)!n!}$.

Proof outline (thus this answer is very incomplete):

$\frac{4^0}{(1)!0!} = 1 = a_0$

$a_n = \frac{4a_{n-1}}{(n+1)n} = \frac{4(4^{n-1})}{n!(n-1)!(n+1)n} = \frac{4^n}{(n+1)!n!}$

7.) Given the differential equation: $xy'' + 4y = 0$

[3] 7a.) Show that $x = 0$ is a regular singular point of the above differential equation.

[4] 7b.) The indicial equation is ________________________

[1] 7c.) The roots are the indicial equation are ________________________

[5] 7d.) After plugging in the larger root, the recurrence relation is ________________________

[3] 7e.) For the larger root, if $a_0 = 1$, then $a_1 =$ ________, $a_2 =$ ________, $a_3 =$ ________.

[4] 7f.) Use your answer to 5e to give an approximation of one solution to the above differential equation.

Answer for 7f: ____

Let $y = \sum_{n=0}^{\infty} a_n x^{n+r}$

$$\begin{aligned}
xy'' + 4y &= x\sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r-2} + 4\sum_{n=0}^{\infty} a_n x^{n+r} \\
&= \sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r-1} + \sum_{n=0}^{\infty} 4a_n x^{n+r} \\
&= \sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r-1} + \sum_{n=1}^{\infty} 4a_{n-1} x^{n+r-1} \\
&= a_0 r(r-1)x^{r-1} + \sum_{n=1}^{\infty} a_n(n+r)(n+r-1)x^{n+r-1} + \sum_{n=1}^{\infty} 4a_{n-1} x^{n+r-1} \\
&= a_0 r(r-1)x^{r-1} + \sum_{n=1}^{\infty} [a_n(n+r)(n+r-1) + 4a_{n-1}]x^{n+r-1}
\end{aligned}$$

$r(r-1) = 0$ implies $r = 0, 1$

$a_n(n+r)(n+r-1) + 4a_{n-1} = 0$ implies $a_n = \frac{-4a_{n-1}}{(n+r)(n+r-1)}$

Sidenote: When $r = 0$, $n = 1$: $a_1(0) + 4a_0 = 0$ implies $a_0 = 0$, a contradiction.

When $r = 1$, $a_n = \frac{4a_{n-1}}{(n+1)n}$

$n = 1$: $\quad a_1 = \frac{-4a_0}{2} = -2a_0$

$n = 2$: $\quad a_2 = \frac{-4a_1}{(3)2} = \frac{4^2 a_0}{3!2} = \frac{4a_0}{3}$

$n = 3$: $\quad a_3 = \frac{-4a_2}{(4)3} = \frac{-4^3 a_0}{(4)3(2)3!} = \frac{-4^3 a_0}{4!3!} = \frac{-4^2 a_0}{3!3!} = \frac{-4a_0}{9}$

$n = 4$: $\quad a_4 = \frac{-4a_3}{(5)4} = \frac{4^4 a_0}{(5)4(4!3!)} = \frac{4^4 a_0}{5!4!}$