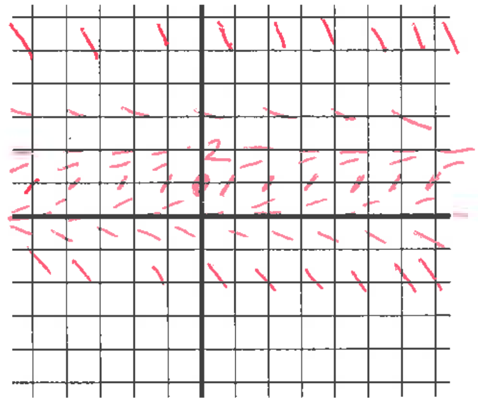
Math 3600 Differential Equations Exam #1
Sept 26, 2018 SHOW ALL WORK

[10] 1a.) Draw the direction field for the following differential equation: $y' = y(2-y)$



[4] 1b.) On the direction field above, draw the solution to the above differential equation that satisfies the initial condition $y(0) = 1$.

Note the solution satisfying the initial condition $y(0) = 1$ must pass thru the initial value (0, 1).

[6] 1c.) Does the differential equation whose direction field is given above have any equilibrium solutions? If so, state whether they are stable, semi-stable or unstable.

An equilibrium solution is a constant solution, $y = C$.

The graph of a constant solution, $y = C$ is a horizontal line.

$y = 2$ is stable and $y = 0$ is unstable

[5] 2.) Give an example of an initial value problem that does not have a unique solution.

The classic example is $y^{\frac{1}{3}}, y(0) = 0$, which as an infinite number of solutions.

3.) Circle T for true and F for false.

[5] 2c.) Suppose $y = \phi_1(t)$ and $y = \phi_2(t)$ are solutions to $ay'' + by' + cy = 0$. Then $y = c_1\phi_1(t) + c_2\phi_2(t)$ is also a solution to this linear homogeneous differential equation.

T

[5] 2d.) Suppose $y = \phi_1(t)$ and $y = \phi_2(t)$ are linearly independent solutions to $ay'' + by' + cy = 0$. If $y = h(t)$ is also a solution to $ay'' + by' + cy = 0$, then there exists constants c_1 and c_2 such that $h(t) = c_1\phi_1(t) + c_2\phi_2(t)$.

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[20] 4.) Find the general solution to $ty' - 2y = t^3e^t - 8$. Also find the solution that passes thru the point (1, 3). How does the solution passing thru (1, 3) behave as $t \to \infty$?

$ty' - 2y = t^3e^t - 8$ implies $1y' + (-\frac{2}{t})y = t^2e^t - 8t^{-1}$.

Integrating factor: $u = e^{\int p(t)dt} = e^{\int -\frac{2}{t}dt} = e^{-2ln|t|} = e^{ln(|t|^{-2})} = t^{-2}$

Let $u(t) = t^{-2}$

$t^{-2}y' - 2t^{-3}y = e^t - 8t^{-3}$

$(t^{-2}y)' = e^t - 8t^{-3}$

Check: $(t^{-2}y)' = t^{-2}y' - 2t^{-3}y$

$\int(t^{-2}y)'dt = \int(e^t - 8t^{-3})dt$

$t^{-2}y = e^t + 4t^{-2} + C$

$y = t^2e^t + 4 + Ct^2$

$y(1) = 3$: $\quad 3 = (1)^2e^1 + 4 + C(1)^2 = e + 4 + C$

$3 = e + 4 + C$. Thus $C = -1 - e$

IVP soln: $y = t^2e^t + 4 + (-1 - e)t^2$

$y = t^2e^t + 4 - t^2 - et^2$

$y = t^2(e^t - e - 1) + 4$

$t \to +\infty$, $t^2 \to +\infty$ and $e^t \to +\infty$ and hence $e^t - e - 1 \to +\infty$.

Thus $y = t^2(e^t - e - 1) + 4 \to +\infty$

General solution: $\underline{\quad y = t^2e^t + 4 + Ct^2 \quad}$

IVP solution: $\underline{\quad y = t^2(e^t - e - 1) + 4 \quad}$

$t \to \infty$, $y \to \underline{\quad +\infty \quad}$

5.) Solve the following 2nd order differential equations

[15] 5a.) $2y'' - y' + 10y = 0$

Guess $y = e^{rt}$

$2r^2 - r + 10 = 0$. Thus $r = \frac{1 \pm \sqrt{(-1)^2 - 4(2)(10)}}{2(2)} = \frac{1 \pm \sqrt{(1-80)}}{4} = \frac{1 \pm \sqrt{-79}}{4} = \frac{1}{4} \pm i\frac{\sqrt{79}}{4}$

General solution: $y = c_1 e^{\frac{t}{4}} cos(\frac{\sqrt{79}}{4}t) + c_2 e^{\frac{t}{4}} sin(\frac{\sqrt{79}}{4}t)$

[15] 5b.) $x(x'') = (x)^2$ Hint: Let $x' = \frac{dx}{dt} = v$, then $v' = \frac{dv}{dt} = \frac{dx}{dt}\frac{dv}{dx} = v\frac{dv}{dx}$

Also $x' = \frac{dx}{dt} = v$ implies $x'' = v' = \frac{dv}{dt} = \frac{dx}{dt}\frac{dv}{dx} = v\frac{dv}{dx}$

$x\frac{dv}{dt} = v^2$, but this has 3 variables, so replace $\frac{dv}{dt} = \frac{dx}{dt}\frac{dv}{dx} = v\frac{dv}{dx}$

$xv\frac{dv}{dx} = v^2$

$\frac{dv}{v} = \frac{dx}{x}$

$ln|v| = ln|x| + C$

$v = Cx$

$\frac{dx}{dt} = Cx$

$\frac{dx}{x} = Cdt$

$ln|x| = Ct + k$

$x = ke^{Ct}$

Check: $(ke^{Ct})(kC^2e^{Ct}) = Cke^{Ct}Cke^{Ct}$

General solution: $x = ke^{Ct}$

[15] 6.) Show by induction that for Picard's iteration method, $\phi_n(t) = \sum_{k=1}^{n} \frac{t^{2k}}{k!}$ approximates the solution to the initial value problem, $y' = 2t(1+y)$, $y(0) = 0$ where $\phi_1(t) = t^2$. You may use the proof outline below or write it from scratch.

Proof by induction on n.

For $n = 1$, $\sum_{k=1}^{1} \frac{t^{2k}}{k!} = \frac{3(-1)^{1+1}\ t^{1+1}}{(1+1)!} = \frac{3\ t^2}{2!} = \phi_1(t)$

Suppose for $n = j$, $\phi_{j-1}(t) = \sum_{k=1}^{j-1} \frac{t^{2k}}{k!}$

Claim: $\phi_j = \sum_{k=1}^{j} \frac{t^{2k}}{k!}$

Proof of claim: By Picard's iteration method, $\phi_j = \int_0^t f(s, \phi_{j-1}(s))ds$

$$= \int_0^t 2s(1 + \sum_{k=1}^{j-1} \frac{s^{2k}}{k!})ds$$

$$= \int_0^t (2s + \sum_{k=1}^{j-1} \frac{2s^{2k+1}}{k!})ds$$

$$= \int_0^t (\sum_{k=0}^{j-1} \frac{2s^{2k+1}}{k!})ds$$

$$= \sum_{k=0}^{j-1} \frac{2t^{2k+2}}{(2k+2)k!}$$

$$= \sum_{k=0}^{j-1} \frac{2t^{2k+2}}{2(k+1)k!}$$

$$= \sum_{k=0}^{j-1} \frac{t^{2k+2}}{(k+1)!}$$

$$= \sum_{k=1}^{j} \frac{t^{2k}}{k!}$$