Math 3600 Differential Equations Exam #2
Oct 26, 2018 SHOW ALL STEPS

1.) Circle T for true and F for false.

[4] 1a.) Suppose $f(x) = \Sigma a_n (x-3)^n$ has a radius of convergence $= r$ about 3. Then we can define the domain of f to be $(3-r, 3+r)$.
T

[4] 1b.) If $b^2 - 4ac < 0$, then the solution to the initial value problem $ay'' + by' + cy = 0$, $y(0) = 2$, $y'(0) = 1$ is a complex valued function.
F

[4] 1c.) If $b^2 - 4ac < 0$, then the solution to the characteristic equation $ar^2 + br + c = 0$ is complex valued.
T

[4] 1d.) $D(f) = f'$ is a linear function.
T

[4] 1e.) There is a unique solution to the differential equation $ay''+by'+cy = g(t), y(0) = 1, y(1) = 0$
F

[7] 2.) The eigenvalues of $\begin{pmatrix} 3 & -2 \\ 1 & 5 \end{pmatrix}$ are $\underline{\ 4 \pm i\ }$

$$\begin{vmatrix} 3-\lambda & -2 \\ 1 & 5-\lambda \end{vmatrix} = (3-\lambda)(5-\lambda) + 2 = 15 - 8\lambda + \lambda^2 + 2 = \lambda^2 - 8\lambda + 17$$

$$\lambda = \frac{8 \pm \sqrt{8^2 - 4(17)}}{2} = \frac{8 \pm 2\sqrt{2(8) - 17}}{2} = 4 \pm \sqrt{-1} = 4 \pm i$$

[7] 3.) Suppose $A\begin{bmatrix} 4 \\ 12 \end{bmatrix} = \begin{bmatrix} -3 \\ 11 \end{bmatrix}$, $A\begin{bmatrix} 1 \\ 7 \end{bmatrix} = \begin{bmatrix} 3 \\ 21 \end{bmatrix}$, $A\begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 9 \\ 31 \end{bmatrix}$, $A\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} -6 \\ -10 \end{bmatrix}$

$$A\begin{bmatrix} 1 \\ 7 \end{bmatrix} = \begin{bmatrix} 3 \\ 21 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 7 \end{bmatrix},$$

$$A\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} -6 \\ -10 \end{bmatrix} = -2\begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

State the 2 eigenvalues of A:

3, -2

State 5 eigenvectors of A:

$\begin{bmatrix} 1 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ 14 \end{bmatrix}, \begin{bmatrix} -1 \\ -7 \end{bmatrix}, \begin{bmatrix} -3 \\ -21 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \end{bmatrix}$, etc.

[20] 4.) Using power series, find a degree 5 polynomial approximation for the solution to $y''-y=4x$ for x near 0

$y=\Sigma_{n=0}^{\infty}a_nx^n$, $y'=\Sigma_{n=1}^{\infty}a_nnx^{n-1}$, $y''=\Sigma_{n=2}^{\infty}a_nn(n-1)x^{n-2}$.

$\Sigma_{n=2}^{\infty}a_nn(n-1)x^{n-2}-\Sigma_{n=0}^{\infty}a_nx^n=4x$

$\Sigma_{n=2}^{\infty}a_nn(n-1)x^{n-2}-\Sigma_{n=0}^{\infty}a_nx^n=4x$

$\Sigma_{n=0}^{\infty}a_{n+2}(n+2)(n+1)x^n-\Sigma_{n=0}^{\infty}a_nx^n=4x$

$\Sigma_{n=0}^{\infty}[a_{n+2}(n+2)(n+1)-a_n]x^n=4x$

For $n=1$: $[a_3(3)(2)-a_1]x=4x$. Thus $a_3=\frac{a_1+4}{6}$

For $n\neq 1$, $a_{n+2}(n+2)(n+1)-a_n=0$. Thus $a_{n+2}=\frac{a_n}{(n+2)(n+1)}$

For $n=0$: $a_2=\frac{a_0}{(2)(1)}$

For $n=2$: $a_4=\frac{a_2}{(4)(3)}=\frac{a_0}{(4)(3)(2)(1)}$

For $n=3$: $a_5=\frac{a_3}{(5)(4)}=\frac{a_1+4}{6(5)(4)}$

$$\text{Approximation: } y=a_0+a_1x+\frac{a_0}{2}x^2+\frac{a_1+4}{6}x^3+\frac{a_0}{4!}x^4+\frac{a_1+4}{120}x^5$$

[22] 5.) Solve $y'' - y = e^t + 2, \quad y(0) = 1,\ y'(0) = 2$

Solve homogeneous: Guess $y = e^{rt}$ and plug into $y'' - y = 0$: $r^2 e^{rt} - e^{rt} = 0$.

Thus $r^2 - 1 = (r+1)(r-1) = 0$. Thus $r = 1, -1$.

Homogeneous solution: $c_1 e^t + c_2 e^{-t}$

Solve $y'' - y = e^t$

$y = e^t$ is a homogeneous solution, so guess $y = Ate^t$. Then $y' = Ae^t + Ate^t$ and $y'' = Ae^t + Ae^t + Ate^t = 2Ae^t + Ate^t$.

Plug into $y'' - y = e^t$:

$2Ae^t + Ate^t - Ate^t = e^t$ implies $2Ae^t = e^t$. Thus $2A = 1$ and $A = \frac{1}{2}$.

Thus $y = \frac{1}{2}te^t$ is one solution to $y'' - y = e^t$

Solve $y'' - y = 2$

Guess $y = B$, then $y' = 0, y'' = 0$.

Plug in: $0 - B = 2$. Thus $B = -2$.

Thus $y = -2$ is one solution to $y'' - y = 2$

Hence general solution to $y'' - y = e^t + 2$ is $y = c_1 e^t + c_2 e^{-t} + \frac{1}{2}te^t - 2$

Solve IVP: $y(0) = 1,\ y'(0) = 2$.

$y = c_1 e^t + c_2 e^{-t} + \frac{1}{2}te^t - 2$

$y' = c_1 e^t - c_2 e^{-t} + \frac{1}{2}te^t + \frac{1}{2}e^t$

$y(0) = 1$: $1 = c_1 + c_2 - 2$ implies $3 = c_1 + c_2$

$y'(0) = 2$: $2 = c_1 - c_2 + \frac{1}{2}$ implies $\frac{3}{2} = c_1 - c_2$

Add equations: $\frac{9}{2} = 2c_1$. Thus $c_1 = \frac{9}{4}$

Subtract equations: $\frac{3}{2} = 2c_2$. Thus $c_2 = \frac{3}{4}$

Solution: $\underline{\quad y = \frac{9}{4}e^t + \frac{3}{4}e^{-t} + \frac{1}{2}te^t - 2 \quad}$

[24] 6.) Solve **two** of the following (from this page and the next page). If you solve all 4, I will grade your best 2 and will give 1 (or 2) points extra credit for 3 (or 4) correct problems):

6a.) If $y = \psi(t)$ is a solution to $py'' + qy' + ry = g(t)$, show that $y = 2\psi(t)$ is a solution to $py'' + qy' + ry = 2g(t)$. Hint use linearity OR plug in.

Using linearity: Recall that $L(y) = py'' + qy' + ry$ is a linear function.

Since $y = \psi(t)$ is a solution to $py'' + qy' + ry = g(t)$, $L(\psi(t)) = g(t)$. Since L is a linear function, $L(2\psi(t)) = 2L(\psi(t)) = 2g(t)$. Thus $y = 2\psi(t)$ is a solution to $py'' + qy' + ry = 2g(t)$.

Plugging in: Since $y = \psi(t)$ is a solution to $py'' + qy' + ry = g(t)$, $p\psi''(t) + q\psi'(t) + r\psi(t) = g(t)$.

Thus $p[2\psi''(t)] + q[2\psi'(t)] + r[2\psi(t)] = 2[p\psi''(t) + q\psi'(t) + r\psi(t)] = 2g(t)$.

Thus $y = 2\psi(t)$ is a solution to $py'' + qy' + ry = 2g(t)$.

6b.) Use your work in problem 5 to solve $y'' - y = 3e^t + 10$ for the general solution.

Homogeneous solution: $c_1e^t + c_2e^{-t}$

Since $y = \frac{1}{2}te^t$ is one solution to $y'' - y = e^t$, $y = \frac{3}{2}te^t$ is one solution to $y'' - y = 3e^t$

Since $y = -2$ is one solution to $y'' - y = 2$, $y = -10$ is one solution to $y'' - y = 10$

Thus general solution to $y'' - y = 3e^t + 10$ is $y = c_1e^t + c_2e^{-t} + \frac{3}{2}te^t - 10$

6c.) Given a_0, a_1 and $a_{n+2} = 2a_{n+1} - a_n$, determine a_n in terms of a_0 and a_1.

$a_2 = 2a_1 - a_0$

$a_3 = 2a_2 - a_1 = 2(2a_1 - a_0) - a_1 = 3a_1 - 2a_0$

$a_4 = 2a_3 - a_2 = 2(3a_1 - 2a_0) - (2a_1 - a_0) = 4a_1 - 3a_0$

$a_5 = 2a_4 - a_3 = 2(4a_1 - 3a_0) - (3a_1 - 2a_0) = 5a_1 - 4a_0$

Answer: $a_n = na_1 - (n-1)a_0$

6d.) Use the ratio test to determine the radius of convergence for the power series $\Sigma_{n=0}^{\infty}\frac{3^n}{2n-1}x^n$. For what values of x does this series converge?

$$\lim_{n\to\infty}\left|\left(\frac{3^{n+1}x^{n+1}}{2(n+1)-1}\right)\left(\frac{2n-1}{3^nx^n}\right)\right| = \lim_{n\to\infty}\left|\frac{3(2n-1)x}{2(n+1)-1}\right| = \lim_{n\to\infty}\left|\frac{3(2n-1)x}{2n+1}\right| = |3x|\lim_{n\to\infty}\frac{2n-1}{2n+1} = |3x| < 1$$

Thus $|x| < \frac{1}{3}$. Thus radius of convergence is $\frac{1}{3}$ and the series converges for all $x \in (-\frac{1}{3}, \frac{1}{3})$ and the series diverges if $|x| > \frac{1}{3}$

If $x = \frac{1}{3}$: $\Sigma_{n=0}^{\infty}\frac{3^n}{2n-1}x^n = \Sigma_{n=0}^{\infty}\frac{3^n}{2n-1}(\frac{1}{3})^n = \Sigma_{n=0}^{\infty}\frac{1}{2n-1} > \Sigma_{n=0}^{\infty}\frac{1}{2n} = \frac{1}{n}\Sigma_{n=0}^{\infty}\frac{1}{n}$

Since $\Sigma_{n=0}^{\infty}\frac{1}{n}$ diverges, $\Sigma_{n=0}^{\infty}\frac{3^n}{2n-1}(\frac{1}{3})^n$ diverges.

If $x = -\frac{1}{3}$: $\Sigma_{n=0}^{\infty}\frac{3^n}{2n-1}x^n = \Sigma_{n=0}^{\infty}\frac{3^n}{2n-1}(-\frac{1}{3})^n = \Sigma_{n=0}^{\infty}\frac{(-1)^n}{2n-1}$. Since $\lim_{n\to\infty}\frac{1}{2n-1} = 0$ and $\frac{1}{2n-1}$ is a decreasing sequence, $\Sigma_{n=0}^{\infty}\frac{(-1)^n}{2n-1}$ converges by the alternating series test.

Thus the series $\Sigma_{n=0}^{\infty}\frac{3^n}{2n-1}x^n$ converges for all $x \in [-\frac{1}{3}, \frac{1}{3})$.