Non linear eqns are sometimes nice but sol'ns to IVP don't always exists and solns to IVP are not always unique

$y' = y^{1/3}$

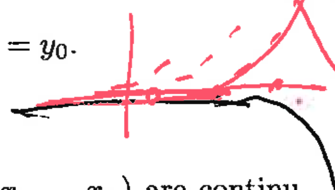
$y(t_0) = 0$

$\Downarrow$

∞ # of soln.

Thm 2.4.2: Suppose the functions $z = f(t, y)$ and $z = \frac{\partial f}{\partial y}(t, y)$ are cont. on $(a, b) \times (c, d)$ and the point $(t_0, y_0) \in (a, b) \times (c, d)$, then there exists an interval $(t_0 - h, t_0 + h) \subset (a, b)$ such that there exists a unique function $y = \phi(t)$ defined on $(t_0 - h, t_0 + h)$ that satisfies the following initial value problem:

$$y' = f(t, y), \quad y(t_0) = y_0.$$



Thm 7.1.1: Suppose the functions $z = F_i(t, x_1, ..., x_n)$ and $z = \frac{\partial F_i}{\partial x_j}(t, x_1, ..., x_n)$ are continuous for all i, j in a region $R = \{(t, x_1, ..., x_n) \mid a < t < b, a_1 < x_1 < b_1, ..., a_n < x_n < b_n\}$, and let the point $(t_0, x_1^0, ..., x_n^0) \in R$. Then there exists an interval $(t_0 - h, t_0 + h) \subset (a, b)$ such that there exists a unique solution defined on $(t_0 - h, t_0 + h)$,

$$x_1 = \phi_1(t), \; ..., \; x_n = \phi_n(t)$$

that satisfies the following initial value problem:

$$\begin{aligned} x_1' &= F_1(t, x_1, ..., x_n) \\ x_2' &= F_2(t, x_1, ..., x_n) \\ &\vdots \\ x_n' &= F_n(t, x_1, ..., x_n) \end{aligned}$$

$$x_1(t_0) = x_1^0, x_2(t_0) = x_2^0, , ..., x_n(t_0) = x_n^0$$

Linear eqns are nice

p_i's cont $\Rightarrow$ unique soln exists for IVP

Theorem 4.1.1: If $p_i : (a, b) \to R$, $i = 1, ..., n$ and $g : (a, b) \to R$ are continuous and $a < t_0 < b$, then there exists a unique function $y = \phi(t)$, $\phi : (a, b) \to R$ that satisfies the initial value problem

$$y^{(n)} + p_1(t)y^{(n-1)} + ... + p_{n-1}(t)y' + p_n(t)y = g(t),$$

$$y(t_0) = y_0, \quad y'(t_0) = y_1, \quad y^{(n-1)}(t_0) = y_{n-1}$$

Thm 7.1.2: If p_{ij} and g_i are continuous on (a, b) and the point $t_0 \in (a, b)$, then there exists a unique solution $x_1 = \phi_1(t)$, ..., $x_n = \phi_n(t)$ defined on (a, b) that satisfies the following initial value problem:

$$\begin{pmatrix} x_1' \\ \cdot \\ \cdot \\ \cdot \\ x_n' \end{pmatrix} = \begin{pmatrix} p_{11}(t) & p_{12}(t) & \dots & p_{1n}(t) \\ p_{21}(t) & p_{22}(t) & \dots & p_{2n}(t) \\ & & \vdots & \\ p_{n1}(t) & p_{n2}(t) & \dots & p_{nn}(t) \end{pmatrix} \begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix} + \begin{pmatrix} g_1(t) \\ \cdot \\ \cdot \\ \cdot \\ g_n(t) \end{pmatrix}$$

$$x_1(t_0) = x_1^0, x_2(t_0) = x_2^0, , ..., x_n(t_0) = x_n^0$$

Thm 7.4.1: If $\mathbf{f_k}(t)$ are solutions to $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ where $P_{ij}(t) = p_{ij}(t)$, then the linear combination $\sum_{i=1}^{k} c_i \mathbf{f_k}(t)$ is also a solution for any constants c_i.

Thm 7.4.2: If $\mathbf{f_1}, ..., \mathbf{f_n}$ are linearly independent solutions to $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ on (a, b), then if $\mathbf{x} = \mathbf{g}(t)$ is also a solution to this equation, then $\mathbf{g}(t) = \sum_{i=1}^{n} c_i \mathbf{f_k}(t)$ for some constants c_i

Solve $\mathbf{X}'(t) = \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} \mathbf{X}(\mathbf{t})$

$\boxed{\textbf{Step 1. Find eigenvalues:}}$

$$A - \lambda I = \begin{vmatrix} 1-\lambda & 3 \\ 4 & 5-\lambda \end{vmatrix} = (1-\lambda)(5-\lambda) - 12$$
$$= \lambda^2 - 6\lambda + 5 - 12 = \lambda^2 - 6\lambda - 7 = (\lambda - 7)(\lambda + 1) = 0$$

Thus $\lambda = 7, -1$ ← unstable saddle

$\boxed{\textbf{Step 2. Find eigenvectors:}}$

$$\lambda = 7: \quad A - 7I = \begin{bmatrix} 1-7 & 3 \\ 4 & 5-7 \end{bmatrix} = \begin{bmatrix} -6 & 3 \\ 4 & -2 \end{bmatrix}$$

Note $\begin{bmatrix} -6 & 3 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Note the dimension of the nullspace of $\begin{bmatrix} -6 & 3 \\ 4 & -2 \end{bmatrix}$ is 1.

Or in other words, solution space for

$$\begin{bmatrix} -6 & 3 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ is 1-dimensional}$$

Thus a basis for the eigenspace for $\lambda = 7$ is $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$

$\hookrightarrow \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{7t}$

$$\lambda = -1 \quad A - (-1)I = \begin{bmatrix} 1+1 & 3 \\ 4 & 5+1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$$

Note $\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 3 \\ -2 \end{bmatrix} e^{-t}$

Thus a basis for the eigenspace for $\lambda = -1$ is $\left\{ \begin{bmatrix} 3 \\ -2 \end{bmatrix} \right\}$

Thus a basis for the solution space to $\mathbf{X}' = \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} \mathbf{X}$ is

$$\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{7t}, \begin{bmatrix} 3 \\ -2 \end{bmatrix} e^{-t} \right\}$$

Hence the general solution is

$$\boxed{\mathbf{X}(t) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{7t} + c_2 \begin{bmatrix} 3 \\ -2 \end{bmatrix} e^{-t}}$$

Note we can take any basis for the solution space to create the general solution

Alternate basis: $\left\{ \begin{bmatrix} 2 \\ 4 \end{bmatrix} e^{7t}, \begin{bmatrix} -9 \\ 6 \end{bmatrix} e^{-t} \right\}$

Alternate format of general solution:

$$\mathbf{X}(t) = c_1 \begin{bmatrix} 2 \\ 4 \end{bmatrix} e^{7t} + c_2 \begin{bmatrix} -9 \\ 6 \end{bmatrix} e^{-t}$$

plug in

IVP: $\mathbf{X}' = \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix}\mathbf{X}, \quad \mathbf{X}(0) = \begin{bmatrix} e \\ f \end{bmatrix}$

$$\begin{bmatrix} e \\ f \end{bmatrix} = c_1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} c_1 + 3c_2 \\ 2c_1 - 2c_2 \end{bmatrix}$$

Solve using any method you like. We will use matrix form:

$$\begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Solution exists if Wronskian evaluated at 0 is not zero.

$$W\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}e^{7t}, \begin{bmatrix} 3 \\ -2 \end{bmatrix}e^{-t}\right) = \begin{vmatrix} e^{7t} & 3e^{-t} \\ 2e^{7t} & -2e^{-t} \end{vmatrix}$$

det $\neq 0$

$$= -2e^{6t} - 6e^{6t} = -8e^{6t} \neq 0$$

Linear Alg

$\Downarrow$

$\exists!$ soln

Fundamental matrix: $\Phi(t) = \begin{bmatrix} e^{7t} & 3e^{-t} \\ 2e^{7t} & -2e^{-t} \end{bmatrix}$

Back to IVP: $\begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

$$\begin{bmatrix} 1 & 3 \\ 2 & -2 \end{bmatrix}^{-1}\begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & -2 \end{bmatrix}^{-1}\begin{bmatrix} & 3 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Thus $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & -2 \end{bmatrix}^{-1}\begin{bmatrix} e \\ f \end{bmatrix}$

$X' = X$ (FYI)

7.7

But I would prefer a fundamental matrix whose inverse is easier to calculate, at least when $t_0 = 0$.

Thus we will find another basis for the solution set to $\mathbf{x}' = A\mathbf{x}$ so that the corresponding fundamental matrix has the property that $\Phi(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, the 2x2 identity matrix.

Step 1: Solve IVP: $\mathbf{x}' = A\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1\begin{bmatrix} 1 \\ 2 \end{bmatrix}e^0 + c_2\begin{bmatrix} 3 \\ -2 \end{bmatrix}e^0 = \begin{bmatrix} 1 & 3 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \text{ implies } \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & -2 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$(-\tfrac{1}{8})\begin{bmatrix} -2 & -3 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ 2 & -2 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{4} & \frac{3}{8} \\ \frac{1}{4} & -\frac{1}{8} \end{bmatrix} \;\&\; \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{3}{8} \\ \frac{1}{4} & -\frac{1}{8} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \end{bmatrix}$$

Thus IVP solution where $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is

$$\mathbf{X}(t) = \tfrac{1}{4}\begin{bmatrix} 1 \\ 2 \end{bmatrix}e^{7t} + \tfrac{1}{4}\begin{bmatrix} 3 \\ -2 \end{bmatrix}e^{-t} = \begin{bmatrix} \frac{1}{4}e^{7t} + \frac{3}{4}e^{-t} \\ \frac{1}{2}e^{7t} - \frac{1}{2}e^{-t} \end{bmatrix}$$

$\exists!$ soln $\Longleftrightarrow$ det $\neq 0$

7.7 (not on final)

Step 2: Solve IVP: $\mathbf{x}' = A\mathbf{x}$, $\mathbf{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^0 + c_2 \begin{bmatrix} 3 \\ -2 \end{bmatrix} e^0 = \begin{bmatrix} 1 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Thus $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{3}{8} \\ \frac{1}{4} & -\frac{1}{8} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{8} \\ -\frac{1}{8} \end{bmatrix}$

Thus IVP solution where $\mathbf{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is

$$\mathbf{X}(t) = \frac{3}{8} \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{7t} - \frac{1}{8} \begin{bmatrix} 3 \\ -2 \end{bmatrix} e^{-t} = \begin{bmatrix} \frac{3}{8}e^{7t} - \frac{3}{8}e^{-t} \\ \frac{3}{4}e^{7t} + \frac{1}{4}e^{-t} \end{bmatrix}$$

Thus another basis for the solution space to $\mathbf{X}' = \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} \mathbf{X}$

is $\left\{ \begin{bmatrix} \frac{1}{4}e^{7t} + \frac{3}{4}e^{-t} \\ \frac{1}{2}e^{7t} - \frac{1}{2}e^{-t} \end{bmatrix}, \begin{bmatrix} \frac{3}{8}e^{7t} - \frac{3}{8}e^{-t} \\ \frac{3}{4}e^{7t} + \frac{1}{4}e^{-t} \end{bmatrix} \right\}$

Its corresponding fundamental matrix is

$$\begin{bmatrix} \frac{1}{4}e^{7t} + \frac{3}{4}e^{-t} & \frac{3}{8}e^{7t} - \frac{3}{8}e^{-t} \\ \frac{1}{2}e^{7t} - \frac{1}{2}e^{-t} & \frac{3}{4}e^{7t} + \frac{1}{4}e^{-t} \end{bmatrix}$$

Thus to solve IVP where $\mathbf{X}(t_0) = \begin{bmatrix} e \\ f \end{bmatrix}$, we solve

$$\begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} \frac{1}{4}e^{7t_0} + \frac{3}{4}e^{-t_0} & \frac{3}{8}e^{7t_0} - \frac{3}{8}e^{-t_0} \\ \frac{1}{2}e^{7t_0} - \frac{1}{2}e^{-t_0} & \frac{3}{4}e^{7t_0} + \frac{1}{4}e^{-t_0} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

When $t_0 = 0$. I.e., we have an IVP where $\mathbf{X}(0) = \begin{bmatrix} e \\ f \end{bmatrix}$

$$\begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} \frac{1}{4}e^0 + \frac{3}{4}e^0 & \frac{3}{8}e^0 - \frac{3}{8}e^0 \\ \frac{1}{2}e^0 - \frac{1}{2}e^0 & \frac{3}{4}e^0 + \frac{1}{4}e^0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

which simplifies to

$$\begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

In other words, $c_1 = e$ and $c_2 = f$.