$$= c_1 \begin{bmatrix} v_1 + iw_1 \\ v_2 + iw_2 \end{bmatrix} e^{\lambda t}[cos(\mu t) + isin(\mu t)] + c_2 \begin{bmatrix} v_1 - iw_1 \\ v_2 - iw_2 \end{bmatrix} e^{\lambda t}[cos(-\mu t) + isin(-\mu t)]$$

$$= c_1 \begin{bmatrix} v_1 + iw_1 \\ v_2 + iw_2 \end{bmatrix} e^{\lambda t}[cos(\mu t) + isin(\mu t)] + c_2 \begin{bmatrix} v_1 - iw_1 \\ v_2 - iw_2 \end{bmatrix} e^{\lambda t}[cos(\mu t) - isin(\mu t)]$$

$$= c_1 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} e^{\lambda t}[cos(\mu t) + isin(\mu t)] + c_1 \begin{bmatrix} iw_1 \\ iw_2 \end{bmatrix} e^{\lambda t}[cos(\mu t) + isin(\mu t)]$$
$$+ c_2 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} e^{\lambda t}[cos(\mu t) - isin(\mu t)] - c_2 \begin{bmatrix} iw_1 \\ iw_2 \end{bmatrix} e^{\lambda t}[cos(\mu t) - isin(\mu t)]$$

$$= c_1 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} e^{\lambda t}[cos(\mu t) + isin(\mu t)] + c_1 \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} e^{\lambda t}[icos(\mu t) + i^2 sin(\mu t)]$$
$$+ c_2 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} e^{\lambda t}[cos(\mu t) - isin(\mu t)] - c_2 \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} e^{\lambda t}[icos(\mu t) - i^2 sin(\mu t)]$$

$$= (c_1 + c_2) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} e^{\lambda t} cos(\mu t) + i(c_1 - c_2) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} e^{\lambda t} sin(\mu t)$$
$$+ i(c_1 - c_2) \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} e^{\lambda t} cos(\mu t) - (c_1 + c_2) \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} e^{\lambda t} sin(\mu t)$$

$$= (c_1 + c_2) \left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} cos(\mu t) - \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} sin(\mu t) \right) e^{\lambda t}$$
$$+ i(c_1 - c_2) \left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} sin(\mu t) + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} cos(\mu t) \right) e^{\lambda t}$$

Then general solution is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 \left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} cos(\mu t) - \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} sin(\mu t) \right) e^{\lambda t}$$
$$+ c_2 \left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} sin(\mu t) + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} cos(\mu t) \right) e^{\lambda t}$$

e. value : $\lambda + i\mu$ $\quad \lambda - i\mu$

e. vector $\vec{v} + i\vec{w}$ $\quad \vec{v} - i\vec{w}$

7.6 Special case: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$A - \lambda I = \begin{vmatrix} a - \lambda & b \\ -b & a - \lambda \end{vmatrix} = (a - \lambda)^2 + b^2 = \lambda^2 - 2a\lambda + a^2 + b^2$$

Thus $\lambda = \frac{2a \pm \sqrt{4a^2 - 4(a^2+b^2)}}{2} = \frac{2a \pm \sqrt{-4b^2}}{2} = a \pm bi$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ implies $\begin{matrix} x_1' = ax_1 + bx_2 \\ x_2' = -bx_1 + ax_2 \end{matrix}$

Change to polar coordinates: $r^2 = x_1^2 + x_2^2$ and $tan\theta = \frac{x_2}{x_1}$

Take derivative with respect to t of both equations:

$2rr' = 2x_1x_1' + 2x_2x_2'$ implies

$rr' = x_1(ax_1 + bx_2) + x_2(-bx_1 + ax_2)$

$= ax_1^2 + bx_1x_2 - bx_1x_2 + ax_2^2 = a(x_1^2 + x_2^2) = ar^2$

Thus $rr' = ar^2$ implies $\frac{dr}{dt} = ar$ and thus $r = Ce^{at}$.

$(sec^2\theta)\theta' = \frac{x_1x_2' - x_1'x_2}{x_1^2} = \frac{x_1(-bx_1+ax_2)-(ax_1+bx_2)x_2}{x_1^2}$

$= \frac{-bx_1^2 + ax_1x_2 - ax_1x_2 - bx_2^2}{x_1^2} = \frac{-b(x_1^2+x_2^2)}{x_1^2} = \frac{-b(r^2)}{x_1^2} = -bsec^2\theta$

$(sec^2\theta)\theta' = -bsec^2\theta$ implies $\theta' = -b$ and thus $\theta = -bt + \theta_0$

Change of basis: Let $\mathbf{x} = P\mathbf{y}$. If $\mathbf{x}' = A\mathbf{x}$, then
$[P\mathbf{y}]' = AP\mathbf{y}$ implies $P\mathbf{y}' = AP\mathbf{y}$. Thus $\mathbf{y}' = P^{-1}AP\mathbf{y}$.