but HW problems in Ch 2 related in Ch 2

Q3

From ICON:

Exam 1 will be this coming Wednesday in class. Please bring your ID. No calculators allowed. The exam will cover the sections covered so far in Ch 1, 2, 3 per course schedule http://homepage.divms.uiowa.edu/~idarcy/COURSES/100/FALL18/3600.html

It will include 1 proof question from the following list.

- Prove a function is not 1:1.
- Prove a function is 1:1.
- Prove a function is not linear.
- Prove a function is linear.
- Show that if y = f(t) and y = g(t) are solutions to ay'' + by' + cy = 0, then y = rf(t) + sg(t) is also a solution to this second order linear homogeneous differential equation.
- 2.8 induction proof.

Copies of exams from previous semesters including answers can be found below:

If 2 real solns r_1 & r_2

$y = c_1(e^{r_1 t}) + c_2(e^{r_2 t})$

The following online book contains many nice examples and good explanations: Paul's Online Notes: Differential Equations

2 complex soln $r = a \pm bi$

$y = c_1(e^{at}\cos(bt)) + c_2(e^{at}\sin(bt))$

1 repeated root $r_1 \Rightarrow y = c_1(e^{r_1 t}) + c_2(te^{r_1 t})$

Note: You must be able to identify which techniques you need to use. For example:

Integration:

* Integration by substitution

* Integration by parts

* Integration by partial fractions

Note: Partial fractions are also used in ch 6 for a different application.

For differential equations:

Is the differential equation 1rst order or 2nd order?

If 2nd order: Section 3.1, solve $ay'' + by' + cy = 0$.

Guess $y = e^{rt}$.
$ar^2e^{rt} + bre^{rt} + ce^{rt} = 0$ implies $ar^2 + br + c = 0$,
Need to have two independent solutions.

If $y = \phi_1, y = \phi_2$ are solutions to a LINEAR HOMOGENEC differential equation, $y = c_1\phi_1 + c_2\phi_2$ is also a solution

If 1st order: Is the equation linear or separable or ?

IVP: plug in initials to find constants

Solving second order differential equation:

p. 135: $y'' = f(t, y')$, $y'' = f(y, y')$,

Transform to first order: Let $v = y'$.

If needed, note $v' = \frac{dv}{dt} = \frac{dv}{dt}\frac{dy}{dy} = \frac{dv}{dy}\frac{dy}{dt} = \frac{dv}{dy}v$.

Note this trick sometimes helpful for first order equations.

Ch 3: linear $ay'' + by' + cy = 0$,

Need to have two independent solutions.

If ϕ_1, ϕ_2 are solutions to a LINEAR HOMOGENEOUS differential equation, $c_1\phi_1 + c_2\phi_2$ is also a solution

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direction field = slope field = graph of $\frac{dv}{dt}$ in t, v-plane.

*** can use slope field to determine behavior of v including as $t \to \infty$.
Equilibrium Solution = constant solution

stable, unstable, semi-stable.

Solving first order differential equation:

Method 1 (sect. 2.2): Separate variables.

Method 2 (sect. 2.1): If linear $[y'(t)+p(t)y(t) = g(t)]$, multiply equation by an integrating factor $u(t) = e^{\int p(t)dt}$

$$\begin{aligned} y' + py &= g \\ y'u + upy &= ug \\ (uy)' &= ug \\ \int (uy)' &= \int ug \\ uy &= \int ug \\ &\text{etc...} \end{aligned}$$

Method 3 (sect. 2.4): Solve Bernoulli's equation,

$$y' + p(t)y = g(t)y^n,$$

when $n > 1$ by changing it to a linear equation by substituting $v = y^{1-n}$

If $v = \frac{dx}{dt}$, can use the following to simplify (especially if there are 3 variables).

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$$

2.3 #22

Short Method:

$m\frac{dv}{dt} = -mg - \frac{v^2}{1325}$

$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = \frac{dv}{dx}v$

$mv\frac{dv}{dx} = -mg - \frac{v^2}{1325}$

$\frac{mvdv}{-mg-\frac{v^2}{1325}} = dx$

Let $u = -mg - \frac{v^2}{1325}$, $du = -\frac{2vdv}{1325}$,

$-\frac{1325mdu}{2u} = dx$

$-\int \frac{1325mdu}{2u} = \int dx$

$-\frac{1325m}{2}ln|u| = x + C$

$-\frac{1325m}{2}ln| - mg - \frac{v^2}{1325}| = x + C$

$-\frac{1325m}{2}ln|mg + \frac{v^2}{1325}| = x + C$

$x = 0, v = 20$

$-\frac{1325m}{2}ln|mg + \frac{20^2}{1325}| = C$

$> -1325 * .15/2 * ln(.15 * 9.8 + 20^2/1325);$

-56.84696820

$C = -56.84696820$

$v = 0, x = ?$

$-\frac{1325m}{2}ln|mg| - C = x$

$> -1325 * .15/2 * ln(.15 * 9.8);$

-38.28545108

$> -38.28545108 - (-56.84696820);$

18.56151712

Answer: 30 + 18.56151563 = 48.562 m

Long Method:

$ma = F = -mg - \frac{v^2}{1325}$, $m = 0.15$, $g = 9.8$

$m\frac{dv}{dt} = -mg - \frac{v^2}{1325}$

$\frac{dv}{dt} = -g(1 + \frac{v^2}{1325mg})$

$\frac{dv}{1+\frac{v^2}{1325mg}} = -gdt$

$\int \frac{dv}{1+\frac{v^2}{1325mg}} = -g \int dt$

Let $u = \frac{v}{\sqrt{1325mg}}$, $du = \frac{dv}{\sqrt{1325mg}}$, $\sqrt{1325mg}\ du = dv$

$\int \frac{\sqrt{1325mg}du}{1+u^2} = -gt + C$

$\sqrt{1325mg}\ tan^{-1}(u) = -gt + C$

$\sqrt{1325mg}\ tan^{-1}(\frac{v}{\sqrt{1325mg}}) = -gt + C$

$t = 0, v = 20$

$\sqrt{1325mg}\ tan^{-1}(\frac{20}{\sqrt{1325mg}}) = C$

$> sqrt(1325 * .15 * 9.8);$

44.13332075

$> 44.1333 * arctan(20/44.1333);$

18.77823743

C = 18.77823743

$v = 0, t = ?$

$\sqrt{1325mg}\ tan^{-1}(\frac{0}{\sqrt{1325mg}}) = -gt + C$

$0 = -gt + C$, $gt = C$, $t = \frac{C}{g}$

$> 18.77823743/9.8;$

1.916146677

$t = 1.916146677$

$\sqrt{1325mg}\ tan^{-1}(\frac{v}{\sqrt{1325mg}}) = -gt + 18.77823743$

$tan^{-1}(\frac{v}{\sqrt{1325mg}}) = \frac{-gt+18.77823743}{44.13332075}$

$\frac{v}{\sqrt{1325mg}} = tan(\frac{-gt+18.77823743}{44.13332075})$

$v = 44.13332075\ tan(\frac{-gt+18.77823743}{44.13332075})$

$\frac{dx}{dt} = 44.13332075\ tan(\frac{-gt+18.77823743}{44.13332075})$

$dx = 44.13332075\ tan(\frac{-gt+18.77823743}{44.13332075})dt$

$dx = 44.13332075\frac{sin(\frac{-gt+18.77823743}{44.13332075})}{cos(\frac{-gt+18.77823743}{44.13332075})}dt$

Let $u = cos(\frac{-gt+18.77823743}{44.13332075})$

$du = -\frac{g}{44.13332075}sin(\frac{-gt+18.77823743}{44.13332075})dt$

$dx = -\frac{44.13332075^2}{g}\frac{du}{u}$

$\int dx = -\int \frac{44.13332075^2}{g}\frac{du}{u}$

$x = -\frac{44.13332075^2}{g}ln|u| + C$

$x = -\frac{44.13332075^2}{g}ln|cos(\frac{-gt+18.77823743}{44.13332075})| + C$

$t = 0, x = 0(or30)$

$> 44.1333^2/(-9.8) * ln(abs(cos(18.77823743/44.1333)));$

18.56151563

$C = 18.56151563$

$t = 1.91615, x =?$

$x = -\frac{44.13332075^2}{9.8}ln|cos(\frac{-9.8(1.91615)+18.77823743}{44.13332075})| + C$

$> -44.1333^2/(9.8) * ln(abs(cos((-9.8 * 1.91615+ 18.77823743)/44.1333)));$

-0.

$x = 0 + 18.56151563 = 18.56151563$

Answer: 30 + 18.56151563 = 48.562 m

Section 2.4: Existence and Uniqueness.

In general, for $y' = f(t,y)$, $y(t_0) = y_0$, solution may or may not exist and solution may or may not be unique.

But we have 2 theorems that guarantee both existence and uniqueness of solutions under certain conditions:

1st order LINEAR differential equation:

Thm 2.4.1: If $p : (a,b) \to R$ and $g : (a,b) \to R$ are continuous and $a < t_0 < b$, then there exists a unique function $y = \phi(t)$, $\phi : (a,b) \to R$ that satisfies the initial value problem

$$y' + p(t)y = g(t),$$
$$y(t_0) = y_0$$

1st order differential equation (general case):

Thm 2.4.2: Suppose $z = f(t,y)$ and $z = \frac{\partial f}{\partial y}(t,y)$ are continuous on $(a,b) \times (c,d)$ and the point $(t_0, y_0) \in (a,b) \times (c,d)$, then there exists an interval $(t_0 - h, t_0 + h) \subset (a,b)$ such that there exists a unique function $y = \phi(t)$ defined on $(t_0 - h, t_0 + h)$ that satisfies the following initial value problem:

$$y' = f(t,y), \quad y(t_0) = y_0.$$

Note the initial value problem

$$y' = y^{\frac{1}{3}}, \; y(0) = 0$$

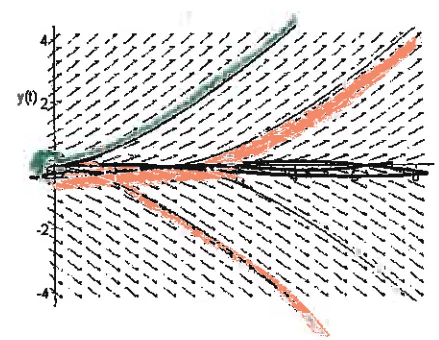
has an infinite number of different solutions.

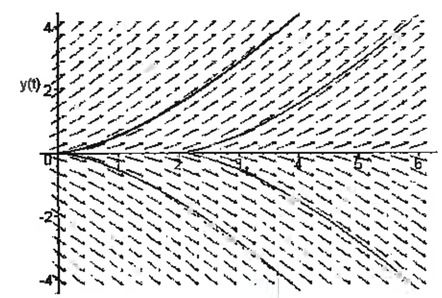
$$y^{-\frac{1}{3}} dy = dt$$
$$\frac{3}{2} y^{\frac{2}{3}} = t + C$$
$$y = \pm(\tfrac{2}{3}t + C)^{\frac{3}{2}}$$
$$y(0) = 0 \text{ implies } C = 0$$

Thus $y = \pm(\frac{2}{3}t)^{\frac{3}{2}}$ are solutions.

$y = 0$ is also a solution, etc.

y' = y^{1/3}





Compare to Thm 2.4.2:
$f(t,y) = y^{\frac{1}{3}}$ is continuous near (0, 0)
But $\frac{\partial f}{\partial y}(t,y) = \frac{1}{3} y^{\frac{-2}{3}}$ is not continuous near (0, 0) since it isn't defined at (0, 0).

Section 2.4 example: $\frac{dy}{dt} = \frac{1}{(1-t)(2-y)}$

$F(y,t) = \frac{1}{(1-t)(2-y)}$ is continuous for all $t \neq 1$, $y \neq 2$

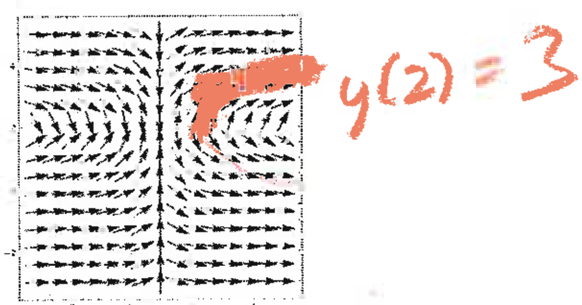
$$\frac{\partial F}{\partial y} = \frac{\partial\left(\frac{1}{(1-t)(2-y)}\right)}{\partial y} = \frac{1}{(1-t)}\frac{\partial(2-y)^{-1}}{\partial y} = \frac{1}{(1-t)(2-y)^2}$$

$\frac{\partial F}{\partial y}$ is continuous for all $t \neq 1$, $y \neq 2$

Thus the IVP $\frac{dy}{dt} = \frac{1}{(1-t)(2-y)}$, $y(t_0) = y_0$ has a unique solution if $t_0 \neq 1$, $y_0 \neq 2$.

Note that if $y_0 = 2$, $\frac{dy}{dt} = \frac{1}{(1-t)(2-y)}$, $y(t_0) = 2$ has two solutions if $t_0 \neq 1$ (and if we allow vertical slope in domain. Note normally our convention will be to NOT allow vertical slope in domain of solution).

Note that if $t_0 = 1$, $\frac{dy}{dt} = \frac{1}{(1-t)(2-y)}$, $y(1) = y_0$ has no solutions.



$(1, 1/((1-t)(2-y)))/sqrt(1 + 1/((1-t)(2-y))^2)$

Solve via separation of variables: $\frac{dy}{dt} = \frac{1}{(1-t)(2-y)}$

$\int (2-y)dy = \int \frac{dt}{1-t}$ implies $2y - \frac{y^2}{2} = -ln|1-t| + C$

$y^2 - 4y - 2ln|1-t| + C = 0$

$$y = \frac{4 \pm \sqrt{16 + 4(2ln|1-t| + C)}}{2} = 2 \pm \sqrt{4 + 2ln|1-t| + C}$$

$$y = 2 \pm \sqrt{2ln|1-t| + C}$$

Find domain: $2ln|1-t| + C \geq 0$ & $t \neq 1$ & $y \neq 2$

NOTE: the convention in this class to to choose largest possible connected domain where tangent line to solution is never vertical.

$2ln|1-t| \geq -C$ and $t \neq 1$ and $y \neq 2$ implies

$ln|1-t| > -\frac{C}{2}$ Note: we want to find domain for this C and thus this C can't swallow constants).

$|1-t| > e^{-\frac{C}{2}}$ since e^x is an increasing function.

$1 - t < -e^{-\frac{C}{2}}$ or $1 - t > e^{-\frac{C}{2}}$

Domain: $\begin{cases} t > e^{-\frac{C}{2}} + 1 & \text{if } t_0 > 1 \\ t < -e^{-\frac{C}{2}} + 1 & \text{if } t_0 < 1. \end{cases}$

$y' = f(t, y(t))$

$\phi_n(t) = \int_0^t \phi' = \int_0^t f((s), \phi_{n-1}(s))\,ds$

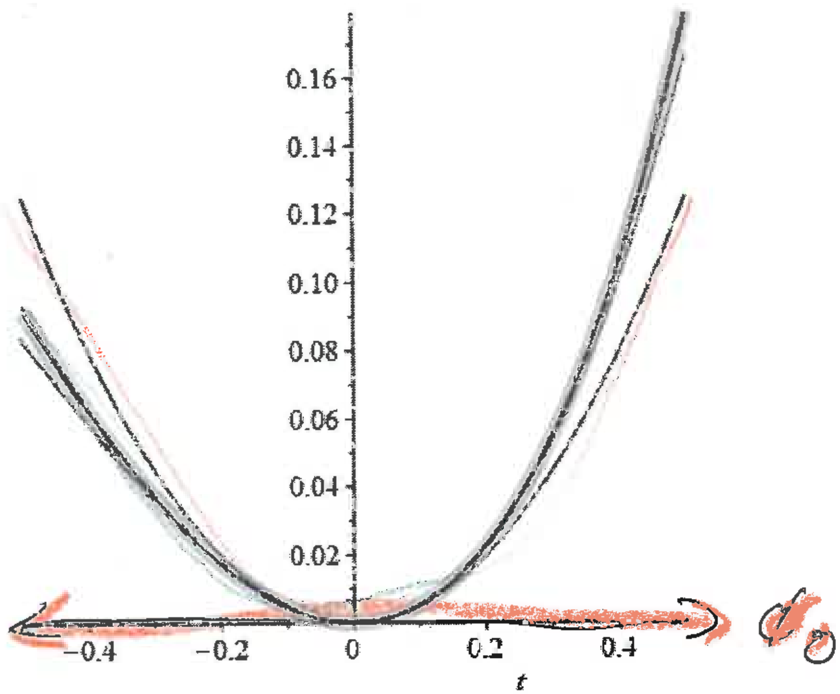
2.8: Approximating soln to IVP using seq of fns,

$$\phi_{n+1}(t) = \int_0^t f(s, \phi_n(s))ds$$

Example: $y' = t + 2y$, $y(0) = 0$

$\phi_0(t) = 0$, $\phi_1(t) = \frac{t^2}{2}$, $\phi_2(t) = \frac{t^2}{2} + \frac{t^3}{3}$,

$\phi_3(t) = \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{6}$, $\phi_4(t) = \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{6} + \frac{t^5}{15}$



Find formula for ϕ_n
look for ! and power
$n!$ 2^n t^n

Proving formula correct for ϕ_n: ~~Induct~~ Induction pf

$n = 1$: $\sum_{n=1}^{1} \sim = \to \to \to \phi_1$

Assume true for $n = m-1$
ie $\phi_{m-1} = \Sigma \sim$

Show true for $n = m$
Claim $\phi_m = \Sigma \sim$

If asked to recognize: I'll give Taylor series

$$\sum_{k=1}^{n} a_k = \sum_{i=0}^{n-1} a_{i+1} = \sum_{k=0}^{n-1} a_{i+1}$$

Let $i = k-1$

$k=1 \Rightarrow i = 1-1 = 0$

$k=n \Rightarrow i = n-1$

$k=k \Rightarrow k = i+1$

$$= \sum_{k=5}^{n+4} a_{k-4}$$

$$\sum_{k=0}^{n} a_k = a_0 + \sum_{k=1}^{n} a_k$$

$$a_0 + \sum_{k=1}^{n} a_k = \sum_{k=0}^{n} a_k$$