Linear comb of solns is also a soln to Linear homog eqn

Summary of sections 3.1, 3, 4: Solve linear homogeneous 2nd order DE with constant coefficients.

Solve $ay'' + by' + cy = 0$. Educated guess $y = e^{rt}$, then

$ar^2e^{rt} + bre^{rt} + ce^{rt} = 0$ implies $\boxed{ar^2 + br + c = 0}$,

Suppose $r = r_1, r_2$ are solutions to $ar^2 + br + c = 0$

$$r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If $r_1 \neq r_2$, then $b^2 - 4ac \neq 0$. Hence a general solution is $y = c_1e^{r_1t} + c_2e^{r_2t}$

If $b^2 - 4ac > 0$, general solution is $\boxed{y = c_1e^{r_1t} + c_2e^{r_2t}}$.

2 real solns roots

If $b^2 - 4ac < 0$, change format to linear combination of real-valued functions instead of complex valued functions by using Euler's formula.

2 complex roots

general solution is $\boxed{y = c_1e^{dt}cos(nt) + c_2e^{dt}sin(nt)}$ where $\boxed{r = d \pm in}$

If $b^2 - 4ac = 0$, $r_1 = r_2$, so need 2nd (independent) solution: te^{r_1t}

Hence general solution is $\boxed{y = c_1e^{r_1t} + c_2te^{r_1t}}$.

1 repeated root

Initial value problem: use $y(t_0) = y_0$, $y'(t_0) = y'_0$ to solve for c_1, c_2 to find unique solution.

slope at (t_0, y_0) is y_0'

Derivation of general solutions:

If $b^2 - 4ac > 0$ we guessed e^{rt} is a solution and noted that any linear combination of solutions is a solution to a homogeneous linear differential equation.

Section 3.3: If $b^2 - 4ac < 0$, :

Changed format of $y = c_1e^{r_1t} + c_2e^{r_2t}$ to linear combination of real-valued functions instead of complex valued functions by using ~~Euler's formula~~:

$$\boxed{e^{it} = cos(t) + isin(t)}$$

Hence $e^{(d+in)t} = e^{dt}e^{int} = e^{dt}[cos(nt) + isin(nt)]$

Let $r_1 = d + in$, $r_2 = d - in$

$y = c_1e^{r_1t} + c_2e^{r_2t} = c_1 e^{(d+in)t} + c_2 e^{(d-in)t}$

$= c_1e^{dt}[cos(nt) + isin(nt)] + c_2e^{dt}[cos(-nt) + isin(-nt)]$

$= c_1e^{dt}cos(nt) + ic_1e^{dt}sin(nt) + c_2e^{dt}cos(nt) - ic_2e^{dt}sin(nt)$

$= (c_1 + c_2)e^{dt}cos(nt) + i(c_1 - c_2)e^{dt}sin(nt)$

$= k_1e^{dt}cos(nt) + k_2e^{dt}sin(nt)$

$c_1 + c_2 \Rightarrow$ constant

$i(c_1 - c_2) \Rightarrow$ constant

Second order differential equation:

Linear equation with constant coefficients:

If the second order differential equation is

$$ay'' + by' + cy = 0,$$

then $y = e^{rt}$ is a solution

Need to have two independent solutions.

Solve the following IVPs:

plug in initial values to solve for c_1 & c_2 (see chalk board)

1.) $y'' - 6y' + 9y = 0$ $\quad$ $y(0) = 1,\ y'(0) = 2$

$r^2 - 6r + 9 = 0$

$(r-3)^2 = 0$

$\Rightarrow r = 3$

General soln

$y = c_1(e^{3t}) + c_2(te^{3t})$

2.) $4y'' - y' + 2y = 0$ $\quad$ $y(0) = 3,\ y'(0) = 4$

$4r^2 - r + 2 = 0$

$r = \frac{1 \pm \sqrt{1 - 4(4)(2)}}{2} = \frac{1}{2} \pm \frac{i\sqrt{31}}{2}$

General soln

$$\boxed{y = c_1 e^{t/2}\cos\left(\tfrac{\sqrt{31}}{2}t\right) + c_2 e^{t/2}\sin\left(\tfrac{\sqrt{31}}{2}t\right)}$$

IVP LTR

3.) $4y'' + 4y' + 1y = 0$ $\quad$ $y(0) = 6,\ y'(0) = 7$

$4r^2 + 4r + 1 = 0$

$(2r+1)(2r+1) = 0$

$\Rightarrow r = -\frac{1}{2}$

General soln

$$\boxed{y = c_1(e^{-t/2}) + c_2(te^{-t/2})}$$

IVP LTR

4.) $2y'' - 2y = 0$ $\quad$ $y(0) = 5,\ y'(0) = 9$

$2r^2 - 2 = 0$

$r^2 - 1 = 0$

$(r-1)(r+1) = 0 \Rightarrow r = \pm 1$

General soln

$$\boxed{y = c_1(e^{t}) + c_2(e^{-t})}$$

IVP LTR

A 1

Solve: $y'' + y = 0$, $y(0) = -1$, $y'(0) = -3$

$y = c_1(\cancel{e^{0t}}\cos(t)) + c_2(\cancel{e^{0t}}\sin(t))$

$r^2 + 1 = 0$ implies $r^2 = -1$. Thus $r = \pm i$.

RECOMMENDED Method:

Since $r = 0 \pm 1i$, $y = k_1 cos(t) + k_2 sin(t)$

Then $y' = -k_1 sin(t) + k_2 cos(t)$

$y(0) = -1$: $-1 = k_1 cos(0) + k_2 sin(0)$ implies $-1 = k_1$

$y'(0) = -3$: $-3 = -k_1 sin(0) + k_2 cos(0)$ implies $-3 = k_2$

$\boxed{\text{Thus IVP solution: } y = -cos(t) - 3sin(t)}$

NOT RECOMMENDED: work with $y = c_1 e^{it} + c_2 e^{-it}$

NOT SIMPLIFIED

$y' = ic_1 e^{it} - ic_2 e^{-it}$

$y(0) = -1$: $-1 = c_1 e^0 + c_2 e^0$ implies $-1 = c_1 + c_2$.

$y'(0) = -3$: $-3 = ic_1 e^0 - ic_2 e^0$ implies $-3 = ic_1 - ic_2$.

$-1i = ic_1 + ic_2$.

$-3 = ic_1 - ic_2$.

$2ic_1 = -3 - i$ implies $c_1 = \frac{-3i - i^2}{-2} = \frac{3i-1}{2}$

$2ic_2 = 3 - i$ implies $c_2 = \frac{3i - i^2}{-2} = \frac{-3i-1}{2}$

Euler's formula: $e^{ix} = cos(x) + isin(x)$

NOT SIMPLIFID

$\boxed{y = (\frac{3i-1}{2})e^{it} + (\frac{-3i-1}{2})e^{-it}} = (\frac{3i-1}{2})[cos(t) + isin(t)] + (\frac{-3i-1}{2})[cos(-t) + isin(-t)]$

$= (\frac{3i-1}{2})[cos(t) + isin(t)] + (\frac{-3i-1}{2})[cos(t) - isin(t)]$

$= (\frac{3i}{2})cos(t) + (\frac{3i}{2})isin(t) + (\frac{-1}{2})cos(t) + (\frac{-1}{2})isin(t) + (\frac{-3i}{2})cos(t) - (\frac{-3i}{2})isin(t) + (\frac{-1}{2})cos(t) - (\frac{-1}{2})isin(t)$

$= (\frac{3i}{2})isin(t) + (\frac{-1}{2})cos(t) + (\frac{3i}{2})isin(t) + (\frac{-1}{2})cos(t)$

$= -(\frac{3}{2})sin(t) - (\frac{1}{2})cos(t) - (\frac{3}{2})sin(t) - (\frac{1}{2})cos(t)$

$\boxed{= -3sin(t) - 1cos(t)}$ simplified :)

So why did we guess $y = e^{rt}$?

Goal: Solve linear homogeneous 2nd order DE with constant coefficients,

$$ay'' + by' + cy = 0 \text{ where } a, b, c \text{ are constants}$$

Standard mathematical technique: make up simpler problems and see if you can generalize to the problem of interest.

Ex: linear homogeneous 1rst order DE: $y' + 2y = 0$

integrating factor $u(t) = e^{\int 2dt} = e^{2t}$

$y'e^{2t} + 2e^{2t}y = 0$

$(e^{2t}y)' = 0$. Thus $\int (e^{2t}y)'dt = \int 0dt$. Hence $e^{2t}y = C$

So $y = Ce^{-2t}$.

Thus exponential function could also be a solution to a linear homogeneous 2nd order DE

$r = -2, r = 0$

$r^2 + 2r = r(r+2)$

Ex: Simple linear homog 2nd order DE $y'' + 2y' = 0$.

no y term $\Rightarrow$ change to 1rst order solve

Let $v = y'$, then $v' = y''$

$y'' + 2y' = 0$ implies $v' + 2v = 0$ implies $v = e^{2t}$.

Thus $v = y' = \frac{dy}{dt} = Ce^{-2t}$. Hence $dy = Ce^{-2t}dt$ and

$$y = c_1 e^{-2t} + c_2.$$

$$\boxed{y = c_1 e^{-2t} + c_2 e^{0t}}$$

plug $v = y'$ back in solve again

$$y = c_1 e^{-2t} + c_2.$$

Note 2 integrations give us 2 constants.

Note also that the general solution is a linear combination of two solutions:

Let $c_1 = 1$, $c_2 = 0$, then we see, $y(t) = e^{-2t}$ is a solution.

Let $c_1 = 0$, $c_2 = 1$, then we see, $y(t) = 1$ is a solution.

The general solution is a linear combination of two solutions:

$$y = c_1 e^{-2t} + c_2(1).$$

Recall: you have seen this before:

Solve linear homogeneous matrix equation $A\mathbf{y} = \mathbf{0}$.

The general solution is a linear combination of linearly independent vectors that span the solution space:

$$\mathbf{y} = c_1\mathbf{v_1} + ... c_n\mathbf{v_n}$$

FYI: You could see this again:

Math 4050: Solve homogeneous linear recurrance relation $x_n - x_{n-1} - x_{n-2} = 0$ where $x_1 = 1$ and $x_2 = 1$.

Fibonacci sequence: $x_n = x_{n-1} + x_{n-2}$

$$1, 1, 2, 3, 5, 8, 13, 21, ...$$

Note $x_n = \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^n - \frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^n$

$y = \phi(t)$ is a soln. Then try $y = v(t)\phi(t)$ ← Method of undetermined in a simplified case

3.6

Section 3.4: If $b^2 - 4ac = 0$, then $r_1 = r_2$.
Hence one solution is $y = e^{r_1 t}$ Need second solution.

If $y = e^{rt}$ is a solution, $y = ce^{rt}$ is a solution.

How about $y = v(t)e^{rt}$?

$y' = v'(t)e^{rt} + v(t)re^{rt}$

$y'' = v''(t)e^{rt} + v'(t)re^{rt} + v'(t)re^{rt} + v(t)r^2e^{rt}$

$\quad = v''(t)e^{rt} + 2v'(t)re^{rt} + v(t)r^2e^{rt}$

$ay'' + by' + cy = 0$

$a(v''e^{rt} + 2v're^{rt} + vr^2e^{rt}) + b(v'e^{rt} + vre^{rt}) + cve^{rt} = 0$

$a(v''(t) + 2v'(t)r + v(t)r^2) + b(v'(t) + v(t)r) + cv(t) = 0$

$av''(t) + 2av'(t)r + av(t)r^2 + bv'(t) + bv(t)r + cv(t) = 0$

$av''(t) + (2ar + b)v'(t) + \cancel{(ar^2 + br + c)v(t)} = 0$ ∵ e^{rt} is a soln

$av''(t) + (\cancel{2}\cancel{a}(\frac{-b}{2\cancel{a}}) + b)v'(t) + 0 = 0$

since $ar^2 + br + c = 0$ and $r = \frac{-b}{2a}$

$av''(t) + (-b + b)v'(t) = 0.$ Thus $av''(t) = 0.$

Hence $v''(t) = 0$ and $v'(t) = k_1$ and $v(t) = k_1t + k_2$

Hence $v(t)e^{r_1t} = (k_1t + k_2)e^{r_1t}$ is a soln

Thus te^{r_1t} is a nice second solution.

Hence general solution is $y = c_1e^{r_1t} + c_2te^{r_1t}$

Solve: $y'' + y = 0$, $y(0) = -1$, $y'(0) = -3$

$r^2 + 1 = 0$ implies $r^2 = -1$. Thus $r = \pm i$.

Since $r = 0 \pm 1i$, $y = k_1cos(t) + k_2sin(t)$.

Then $y' = -k_1sin(t) + k_2cos(t)$

$y(0) = -1$: $-1 = k_1cos(0) + k_2sin(0)$ implies $-1 = k_1$

$y'(0) = -3$: $-3 = -k_1sin(0) + k_2cos(0)$ implies $-3 = k_2$

Thus IVP solution: $y = -cos(t) - 3sin(t)$

When does the following IVP have unique sol'n:

IVP: $ay'' + by' + cy = 0$, $y(t_0) = y_0$, $y'(t_0) = y_1$.

Suppose $y = c_1\phi_1(t) + c_2\phi_2(t)$ is a solution to
$ay'' + by' + cy = 0$. Then $y' = c_1\phi_1'(t) + c_2\phi_2'(t)$

$y(t_0) = y_0$: $y_0 = c_1\phi_1(t_0) + c_2\phi_2(t_0)$

$y'(t_0) = y_1$: $y_1 = c_1\phi_1'(t_0) + c_2\phi_2'(t_0)$

To find IVP solution, need to solve above system of two equations for the unknowns c_1 and c_2.

Note the IVP has a unique solution if and only if the above system of two equations has a unique solution for c_1 and c_2.