Summary of sections 3.1, 3, 4: Solve linear homogeneous 2nd order DE with constant coefficients.

Solve $ay'' + by' + cy = 0$. Educated guess $y = e^{rt}$, then

$ar^2e^{rt} + bre^{rt} + ce^{rt} = 0$ implies $ar^2 + br + c = 0$,

Suppose $r = r_1, r_2$ are solutions to $ar^2 + br + c = 0$

$$r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If $r_1 \neq r_2$, then $b^2 - 4ac \neq 0$. Hence a general solution is $y = c_1e^{r_1t} + c_2e^{r_2t}$

If $b^2 - 4ac > 0$, general solution is $\boxed{y = c_1e^{r_1t} + c_2e^{r_2t}}$.

2 real soln

If $b^2 - 4ac < 0$, change format to linear combination of real-valued functions instead of complex valued functions by using Euler's formula.

2 complex soln

general solution is $\boxed{y = c_1e^{dt}cos(nt) + c_2e^{dt}sin(nt)}$ where $r = d \pm in$

If $b^2 - 4ac = 0$, $r_1 = r_2$, so need 2nd (independent) solution: te^{r_1t}

1 repeat root

Hence general solution is $\boxed{y = c_1e^{r_1t} + c_2te^{r_1t}}$.

Initial value problem: use $y(t_0) = y_0$, $y'(t_0) = y'_0$ to solve for c_1, c_2 to find unique solution.

Examples:

Ex 1: Solve $y'' - 3y' - 4y = 0$, $y(0) = 1$, $y'(0) = 2$.

If $y = e^{rt}$, then $y' = re^{rt}$ and $y'' = r^2e^{rt}$.

$r^2e^{rt} - 3re^{rt} - 4e^{rt} = 0$

$r^2 - 3r - 4 = 0$ implies $(r-4)(r+1) = 0$. Thus $r = 4, -1$

Hence general solution is $y = c_1e^{4t} + c_2e^{-t}$

Solution to IVP:
Need to solve for 2 unknowns, c_1 & c_2; thus need 2 eqns:

$y = c_1e^{4t} + c_2e^{-t}$, $\quad y(0) = 1$ implies $1 = c_1 + c_2$

$y' = 4c_1e^{4t} - c_2e^{-t}$, $\quad y'(0) = 2$ implies $2 = 4c_1 - c_2$

Thus $3 = 5c_1$ & hence $c_1 = \frac{3}{5}$ and $c_2 = 1 - c_1 = 1 - \frac{3}{5} = \frac{2}{5}$

Thus IVP soln: $y = \frac{3}{5}e^{4t} + \frac{2}{5}e^{-t}$

Ex 2: Solve $y'' - 3y' + 4y = 0$.

$y = e^{rt}$ implies $r^2 - 3r + 4 = 0$ and hence

$$r = \frac{3 \pm \sqrt{(-3)^2 - 4(1)(4)}}{2} = \frac{3}{2} \pm \frac{\sqrt{9-16}}{2} = \frac{3}{2} \pm i\frac{\sqrt{7}}{2}$$

Hence general sol'n is $y = c_1e^{\frac{3}{2}t}cos(\frac{\sqrt{7}}{2}t) + c_2e^{\frac{3}{2}t}sin(\frac{\sqrt{7}}{2}t)$

Ex 3: $y'' - 6y' + 9y = 0$ implies $r^2 - 6r + 9 = (r-3)^2 = 0$
Repeated root, $r = 3$ implies
general solution is $y = c_1e^{3t} + c_2te^{3t}$