$h(x) = \cos x$, $h(0) = 1 \neq 0$
$\Rightarrow$ not linear

Linear Functions

A function f is linear if $f(a\mathbf{x} + b\mathbf{y}) = af(\mathbf{x}) + bf(\mathbf{y})$

Or equivalently f is linear if 1.) $f(a\mathbf{x}) = af(\mathbf{x})$ and 2.) $f(\mathbf{x} + \mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y})$

Theorem: If f is linear, then $f(\mathbf{0}) = \mathbf{0}$

Proof: $f(\mathbf{0}) = f(0 \cdot \mathbf{0}) = 0 \cdot f(\mathbf{0}) = \mathbf{0}$

Example 1a.) $f : R \to R$, $f(x) = 2x$

Proof:
$f(ax + by) = 2(ax + by) = 2ax + 2by = af(x) + bf(y)$

Example 1b.) $f : R \to R$, $f(x) = 2x + 3$ is NOT linear.

Proof: $f(2 \cdot 0) = f(0) = 3$, but $2f(0) = 2 \cdot 3 = 6$.
Hence $f(2 \cdot 0) \neq 2f(0)$

Alternate Proof: $f(0 + 1) = f(1) = 5$, but $f(0) + f(1) = 3 + 5 = 8$. Hence $f(0+1) \neq f(0) + f(1)$

Note confusing notation: Most lines, $f(x) = mx + b$ are not linear functions.

Question: When is a line, $f(x) = mx + b$, a linear function?

Example 2.) $f : R^2 \to R^2$,
$f((x_1, x_2)) = (2x_1, x_1 + x_2)$ $f(x) = Ax$

Proof: Let $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2)$

$a\mathbf{x} + b\mathbf{y} = a(x_1, x_2) + b(y_1, y_2) = (ax_1, ax_2) + (by_1, by_2) = (ax_1 + by_1, ax_2 + by_2)$

$f\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$f(ax_1 + by_1, ax_2 + by_2)$

$$= (2(ax_1 + by_1), ax_1 + by_1 + ax_2 + by_2)$$
$$= (2ax_1 + 2by_1, ax_1 + ax_2 + by_1 + by_2)$$
$$= (2ax_1, ax_1 + ax_2) + (2by_1, by_1 + by_2)$$
$$= a(2x_1, x_1 + x_2) + b(2y_1, y_1 + y_2)$$
$$= af((x_1, x_2)) + bf((y_1, y_2))$$

Example 3.) D : set of all differential functions $\to$ set of all functions, $D(f) = f'$

Proof:
$D(af + bg) = (af + bg)' = af' + bg' = aD(f) + bD(g)$

Example 4.) Given a, b real numbers,
I : set of all integrable functions on [a, b] $\to R$,
$I(f) = \int_a^b f$

Proof: $I(sf + tg) = \int_a^b sf + tg = s\int_a^b f + t\int_a^b g = sI(f) + tI(g)$

Example 5.) The inverse of a linear function is linear (when the inverse exists).

Suppose $f^{-1}(x) = c$, $f^{-1}(y) = d$.

Then $f(c) = x$ and $f(d) = y$ and
$f(ac + bd) = af(c) + bf(d) = ax + by$.

Hence $f^{-1}(ax + by) = ac + bd = af^{-1}(x) + bf^{-1}(y)$.

Example 6.) D : set of all twice differential functions $\to$ set of all functions, $L(f) = af'' + bf' + cf$

Proof:

$$L(sf + tg) = a(sf + tg)'' + b(sf + tg)' + c(sf + tg)$$

$$= saf'' + tag'' + sbf' + tbg' + scf + tcg$$

$$= s(af'' + bf' + cf) + t(ag'' + bg' + cg)$$

$$= sL(f) + tL(g)$$

Consequence 1: If ψ_1, ψ_2 are solutions to $af'' + bf' + cf = 0$, then $3\psi_1 + 5\psi_2$ is also a solution to $af'' + bf' + cf = 0$,

Proof: Since ψ_1, ψ_2 are solutions to $af'' + bf' + cf = 0$, $L(\psi_1) = 0$ and $L(\psi_2) = 0$.

Hence $L(3\psi_1 + 5\psi_2) = 3L(\psi_1) + 5L(\psi_2)$

$$= 3(0) + 5(0) = 0.$$

Thus $3\psi_1 + 5\psi_2$ is also a solution to $af'' + bf' + cf = 0$

Consequence 2:
If ψ_1 is a solution to $af'' + bf' + cf = h$
and ψ_2 is a solution to $af'' + bf' + cf = k$,
then $3\psi_1 + 5\psi_2$ is a solution to $af'' + bf' + cf = 3h + 5k$,

Since ψ_1 is a solution to $af'' + bf' + cf = h$, $L(\psi_1) = h$.

Since ψ_2 is a solution to $af'' + bf' + cf = k$, $L(\psi_2) = k$.

Hence $L(3\psi_1 + 5\psi_2) = 3L(\psi_1) + 5L(\psi_2)$

$$= 3h + 5k.$$

Thus $3\psi_1 + 5\psi_2$ is also a solution to
$$af'' + bf' + cf = 3h + 5k$$