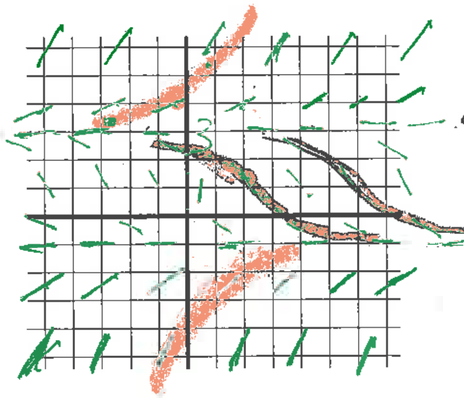
Quiz 1 SHOW ALL WORK

Sept 7, 2018

$y = -1$, $y = 3$

[10] 1.) Draw the direction field for $y' = (y+1)(y-3)$ §2.5

Click on 9/10 date for solution:



unstable

$y = 3$

$y = -1$

stable

[10] 2.) Solve $y' = \frac{1}{x(2y+3)}$, $y(1) = -2$

$\frac{dy}{dx} = \frac{1}{x(2y+3)}$

Compare to 2.2: 14

Separate variables: $(2y+3)dy = \frac{dx}{x}$

$y^2 + 3y = ln|x| + C.$

$y^2 + 3y - ln|x| - C = 0.$

$$y = \frac{-3 \pm \sqrt{9 - 4(-ln|x| - C)}}{2} = \frac{-3 \pm \sqrt{9 + 4ln|x| + C}}{2} = \frac{-3 \pm \sqrt{4ln|x| + C}}{2}$$

General solution: $y = \frac{-3 \pm 2\sqrt{ln|x| + C}}{2}$

IVP: $y(1) = -2$

$$-2 = \frac{-3 \pm 2\sqrt{ln|1| + C}}{2} = \frac{-3 \pm 2\sqrt{C}}{2}$$

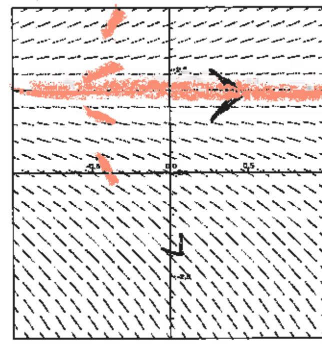
$-4 = -3 \pm 2\sqrt{C}$. Thus $-1 = \pm 2\sqrt{C}$. Hence $-1 = -2\sqrt{C}$.

Thus $\sqrt{C} = \frac{1}{2}$ and $C = \frac{1}{4}$

Hence IVP solution: $y = \frac{-3 - 2\sqrt{ln|x| + \frac{1}{4}}}{2} y = \frac{-3 - 2\sqrt{\frac{4ln|x| + 1}{4}}}{2} = \frac{-3 - \sqrt{4ln|x| + 1}}{2}$

Answer: $y = \frac{-3 - \sqrt{1 + 4ln|x|}}{2}$

2.) Circle the differential equation whose direction field is given below: 1)



pos
y = 2
neg

$y' = f(y)$

A) $\cancel{y' = t^2}$

B) $\cancel{y' = \frac{1}{2}}$

C) $\cancel{y' = 1}$

D) $\cancel{y' = -1}$

E) $y' = y + 1$

F) $y' = y - 2$

G) $y' = (y+1)(y-2)$

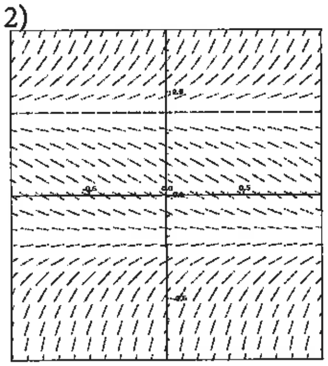
H) $\cancel{y' = (y+1)^2(y-2)^2}$

I) $y' = (y+1)(y-2)^2$

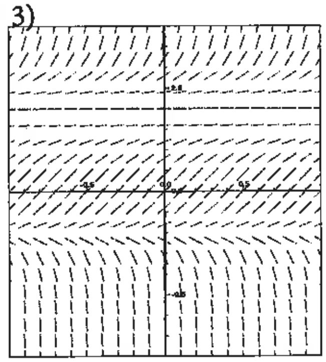
J) $\cancel{y' = (y+1)^2(y-2)}$

K) $\cancel{y = \frac{t}{y}}$

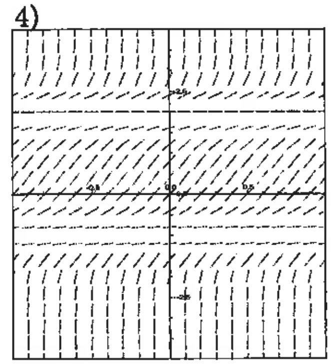
2)



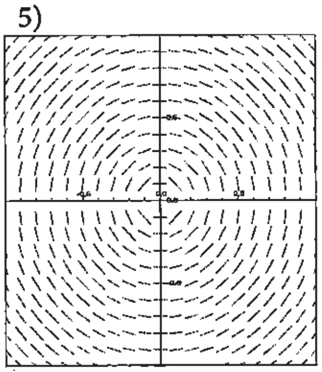
3)



4)



5)



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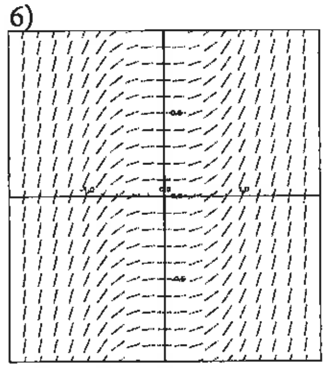
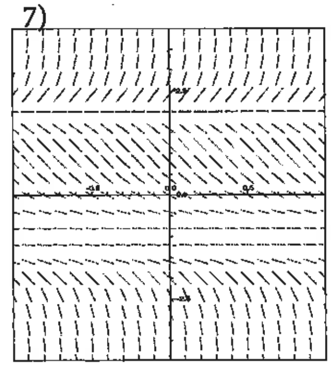


7)



Section 2.4

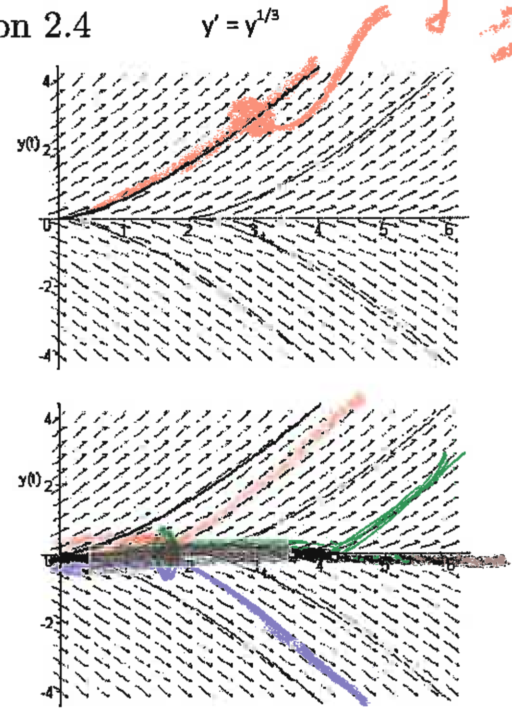


Figure 2.4.1 from ***Elementary Differential Equations and Boundary Value Problems***, Eighth Edition by William E. Boyce and Richard C. DiPrima

$y(t_0) = y_0 \neq 0$
$\Rightarrow$! sol'n

$y(t_0) = 0$

Note IVP, $y' = y^{\frac{1}{3}}$, $y(x_0) = 0$ has an infinite number of solutions,

while IVP, $y' = y^{\frac{1}{3}}$, $y(x_0) = y_0$ where $y_0 \neq 0$ has a unique solution.

Initial Value Problem: $y(t_0) = y_0$
Use initial value to solve for C.

Section 2.4: Existence and Uniqueness.

In general, for $y' = f(t, y)$, $y(t_0) = y_0$, solution may or may not exist and solution may or may not be unique.

Example Non-unique: $y' = y^{\frac{1}{3}}$

$y = 0$ is a solution to $y' = y^{\frac{1}{3}}$ since $y' = 0 = 0^{\frac{1}{3}} = y^{\frac{1}{3}}$

Suppose $y \neq 0$. Then $\frac{dy}{dx} = y^{\frac{1}{3}}$ implies $y^{-\frac{1}{3}} dy = dx$

$\int y^{-\frac{1}{3}} dy = \int dx$ implies $\frac{3}{2} y^{\frac{2}{3}} = x + C$

$y^{\frac{2}{3}} = \frac{2}{3}x + C$ implies $y = \pm\sqrt{(\frac{2}{3}x + C)^3}$

Suppose $y(3) = 0$. Then $0 = \sqrt{(2 + C)^3} \Rightarrow C = -2$.

The IVP, $y' = y^{\frac{1}{3}}$, $y(3) = 0$, has an infinite # of sol'ns

including: $y = 0$, $y = \sqrt{(\frac{2}{3}x - 2)^3}$, $y = -\sqrt{(\frac{2}{3}x - 2)^3}$

Examples: No solution:

Ex 1: $y' = y' + 1$

Ex 2: $(y')^2 = -1$

Ex 3 (IVP): $\frac{dy}{dx} = y(1 + \frac{1}{x})$, $y(0) = 1$

$\int \frac{dy}{y} = \int (1 + \frac{1}{x})dx$ implies $ln|y| = x + ln|x| + C$

$|y| = e^{x + ln|x| + C} = e^x e^{ln|x|} e^C = C|x|e^x = Cxe^x$

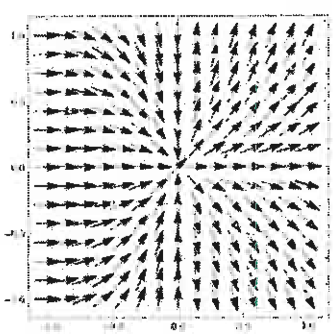
$y = \pm Cxe^x$ implies $y = Cxe^x$

$y(0) = 1$: $1 = C(0)e^0 = 0$ implies

IVP $\frac{dy}{dx} = y(1 + \frac{1}{x})$, $y(0) = 1$ has no solution.

http://www.wolframalpha.com

slope field: $\{1, y(1+1/x)\}/sqrt(1+y\wedge 2(1+1/x)\wedge 2)$



Special cases:

Suppose f is cont. on (a, b) and the point $t_0 \in (a, b)$,
Solve IVP: $\frac{dy}{dt} = f(t)$, $y(t_0) = y_0$

$$dy = f(t)dt$$

$$\int dy = \int f(t)dt$$

$y = F(t) + C$ where F is any anti-derivative of F.

Initial Value Problem (IVP): $y(t_0) = y_0$

$$y_0 = F(t_0) + C \text{ implies } C = y_0 - F(t_0)$$

Hence unique solution (if domain connected) to IVP:

$$y = F(t) + y_0 - F(t_0)$$

First order linear differential equation:

Thm 2.4.1: If p and g are continuous on (a, b) and the point $t_0 \in (a, b)$, then there exists a unique function $y = \phi(t)$ defined on (a, b) that satisfies the following initial value problem:

$$y' + p(t)y = g(t), \quad y(t_0) = y_0.$$

a b cont $\Rightarrow \exists!$ sln $\phi: (a, b) \to \mathbb{R}$
domain could be larger

More general case (but still need hypothesis)

Thm 2.4.2: Suppose the functions
$z = f(t,y)$ and $z = \frac{\partial f}{\partial y}(t,y)$ are continuous on $(a,b) \times (c,d)$ and the point $(t_0, y_0) \in (a,b) \times (c,d)$,

then there exists an interval $(t_0 - h, t_0 + h) \subset (a,b)$ such that there exists a unique function $y = \phi(t)$ defined on $(t_0 - h, t_0 + h)$ that satisfies the following initial value problem:

domain ??

$$y' = f(t,y), \quad y(t_0) = y_0.$$

If possible **without solving**, determine where the solution exists for the following initial value problems:

If not possible **without solving**, state where in the ty-plane, the hypothesis of theorem 2.4.2 is satisfied. In other words, use theorm 2.4.2 to determine where for some interval about t_0, a solution to IVP, $y' = f(t,y)$, $y(t_0) = y_0$ exists and is unique.

Example 1: $ty' - y = 1, \ y(t_0) = y_0$

Example 2: $y' = ln|\frac{t}{y}|, \ y(3) = 6$

Example 3: $(t^2 - 1)y' - \frac{t^3 y}{t-4} = ln|t|, \ y(3) = 6$

Section 2.4 example: $\frac{dy}{dt} = \frac{1}{(1-t)(2-y)}$

$F(y,t) = \frac{1}{(1-t)(2-y)}$ is continuous for all $t \neq 1$, $y \neq 2$

$$\frac{\partial F}{\partial y} = \frac{\partial\left(\frac{1}{(1-t)(2-y)}\right)}{\partial y} = \frac{1}{(1-t)} \frac{\partial (2-y)^{-1}}{\partial y} = \frac{1}{(1-t)(2-y)^2}$$

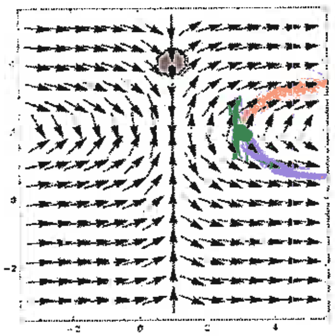
$\frac{\partial F}{\partial y}$ is continuous for all $t \neq 1$, $y \neq 2$

Thus the IVP $\frac{dy}{dt} = \frac{1}{(1-t)(2-y)}$, $y(t_0) = y_0$ has a unique solution if $t_0 \neq 1$, $y_0 \neq 2$.

Note that if $y_0 = 2$, $\frac{dy}{dt} = \frac{1}{(1-t)(2-y)}$, $y(t_0) = 2$ has two solutions if $t_0 \neq 1$ (and if we allow vertical slope in domain. Note normally our convention will be to NOT allow vertical slope in domain of solution).

Note that if $t_0 = 1$, $\frac{dy}{dt} = \frac{1}{(1-t)(2-y)}$, $y(1) = y_0$ has no solutions.

$y(1) = y_0$ ← no sol'n



$y(t_0) = 2$

$(1, 1/((1-t)(2-y)))/sqrt(1+1/((1-t)(2-y))^2)$

Solve via separation of variables: $\frac{dy}{dt} = \frac{1}{(1-t)(2-y)}$

$\int (2-y)dy = \int \frac{dt}{1-t}$

$2y - \frac{y^2}{2} = -ln|1-t| + C$

$y^2 - 4y - 2ln|1-t| + C = 0$

$y = \frac{4 \pm \sqrt{16+4(2ln|1-t|+C)}}{2} = 2 \pm \sqrt{4 + 2ln|1-t| + C}$

$$y = 2 \pm \sqrt{2ln|1-t| + C}$$

Find domain:
$2ln|1-t| + C \geq 0$ and $t \neq 1$ and $y \neq 2$

NOTE: the convention in this class to to choose largest possible connected domain where tangent line to solution is never vertical.

$2ln|1-t| \geq -C$ and $t \neq 1$ and $y \neq 2$ implies

$ln|1-t| > -\frac{C}{2}$ Note: we want to find domain for this C and thus this C can't swallow constants).

$|1-t| > e^{-\frac{C}{2}}$ since e^x is an increasing function.

$1-t < -e^{-\frac{C}{2}}$ or $1-t > e^{-\frac{C}{2}}$

$-t < -e^{-\frac{C}{2}} - 1$ or $-t > e^{-\frac{C}{2}} - 1$

Domain: $\begin{cases} t > e^{-\frac{C}{2}} + 1 & \text{if } t_0 > 1 \\ t < -e^{-\frac{C}{2}} + 1 & \text{if } t_0 < 1. \end{cases}$

Note: Domain is much easier to determine when the ODE is linear.

Find C given $y(t_0) = y_0$: $y_0 = 2 \pm \sqrt{2ln|1-t_0| + C}$

$\pm(y_0 - 2) = \sqrt{2ln|1-t_0| + C}$

$(y_0 - 2)^2 - 2ln|1 - t_0| = C$

$y = 2 \pm \sqrt{2ln|1 - t| + C}$

$y = 2 \pm \sqrt{2ln|1 - t| + (y_0 - 2)^2 - 2ln|1 - t_0|}$

$y = 2 \pm \sqrt{(y_0 - 2)^2 + ln\frac{(1-t)^2}{(1-t_0)^2}}$

Domain: $\begin{cases} t > e^{-\frac{C}{2}} + 1 & \text{if } t_0 > 1 \\ t < -e^{-\frac{C}{2}} + 1 & \text{if } t_0 < 1. \end{cases}$

$e^{-\frac{C}{2}} = e^{-\frac{(y_0-2)^2 - 2ln|1-t_0|}{2}} = |1 - t_0|e^{-\frac{(y_0-2)^2}{2}}$

Domain: $\begin{cases} t > 1 + |1 - t_0|e^{-\frac{(y_0-2)^2}{2}} & \text{if } t_0 > 1 \\ t < 1 - |1 - t_0|e^{-\frac{(y_0-2)^2}{2}} & \text{if } t_0 < 1. \end{cases}$

2.4 #27b. Solve Bernoulli's equation,

$$y' + p(t)y = g(t)y^n,$$

when $n \neq 0, 1$ by changing it

$$y^{-n}y' + p(t)y^{1-n} = g(t)$$

when $n \neq 0, 1$ by changing it to a linear equation by substituting $v = y^{1-n}$

Example: Solve $ty' + 2t^{-2}y = 2t^{-2}y^5$

$$\frac{v'}{(1-n)} + p(t)\,v = g(t)$$

2.4 Solve Bernoulli's equation,

$$y' + p(t)y = g(t)y^n,$$

when $n \neq 0, 1$ by changing it

$$y^{-n}y' + p(t)y^{1-n} = g(t)$$

when $n \neq 0, 1$ by changing it to a linear equation by substituting $v = y^{1-n}$ $\Rightarrow v' = (1-n)\,y^{-n}y'$

Solve $ty' + 2t^{-2}y = 2t^{-2}y^5$

$ty^{-5}y' + 2t^{-2}y^{-4} = 2t^{-2}$ $\leftarrow g(t)$?

Let $v = y^{-4}$. Thus $v' = -4y^{-5}y'$

$-4ty^{-5}y' - 8t^{-2}y^{-4} = -8t^{-2}$

$tv' - 8t^{-2}v = -8t^{-2}$ $\leftarrow$ linear equation

Make coefficient of $v' = 1$

$v' - 8t^{-3}v = -8t^{-3}$

An antiderivative of $-8t^{-3}$ is $4t^{-2}$

Multiply equation by $e^{4t^{-2}}$

$e^{4t^{-2}}v' - 8t^{-3}e^{4t^{-2}}v = -8t^{-3}e^{4t^{-2}}$

$(e^{4t^{-2}}v)' = -8t^{-3}e^{4t^{-2}}$ by PRODUCT rule.

$\int (e^{4t^{-2}}v)'dt = -8\int t^{-3}e^{4t^{-2}}dt$

$e^{4t^{-2}}v = -8\int t^{-3}e^{4t^{-2}}dt.$

Let $u = 4t^{-2}$. Then $du = -8t^{-3}dt$

$e^{4t^{-2}}v = \int e^u du = e^u + C$

$e^{4t^{-2}}v = e^{4t^{-2}} + C$

$v = 1 + Ce^{-4t^{-2}}$

$y^{-4} = 1 + Ce^{-4t^{-2}}$ implies $y = \pm(1 + Ce^{-4t^{-2}})^{-\frac{1}{4}}$

$y' + \frac{2}{t-3}y = 1$

An anti-derivative of $\frac{2}{t-3} = 2ln(t-3)$

$e^{2ln(t-3)} = e^{ln[(t-3)^2]} = (t-3)^2$

$y' + \frac{2}{t-3}y = 1$

$(t-3)^2y' + 2(t-3)y = (t-3)^2$

$\int [(t-3)^2y]' = \int (t-3)^2$

$(t-3)^2y = \frac{(t-3)^3}{3} + C$ implies $y = \frac{(t-3)}{3} + C(t-3)^{-2}$