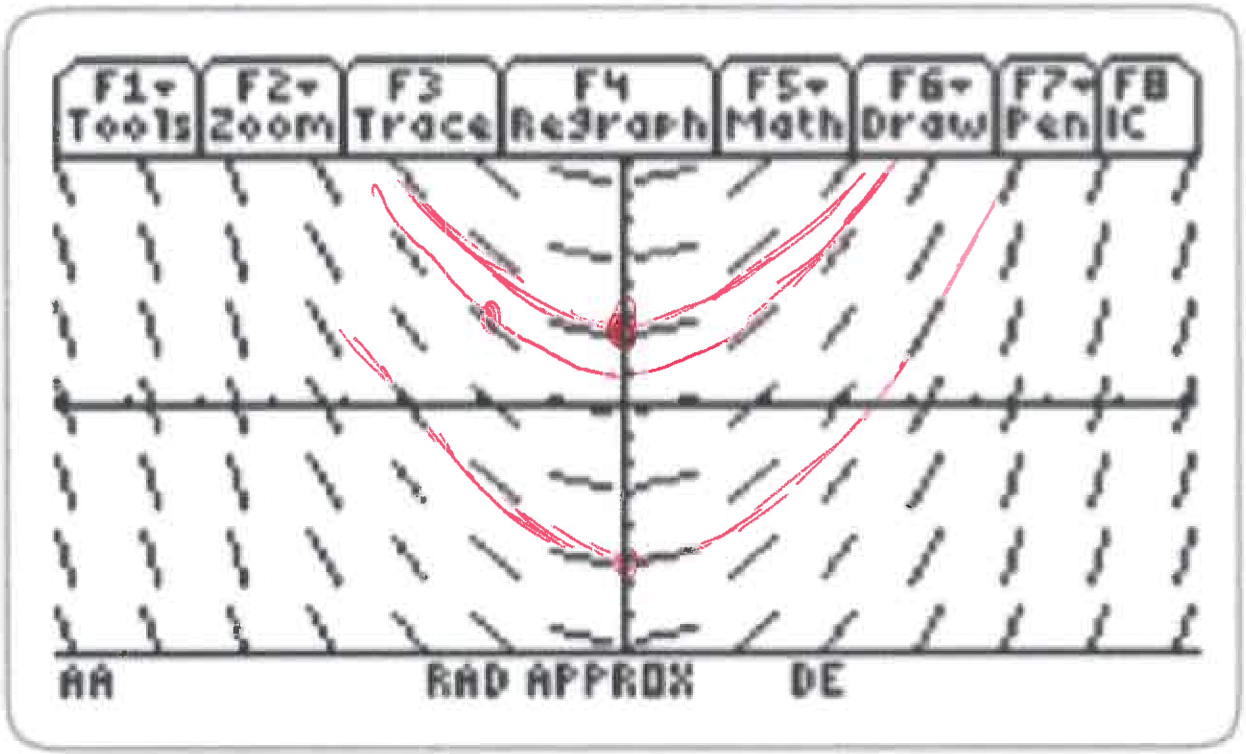
7. Which of the following could be a solution of the differential equation with the given slope field?



(A) ~~$y = x + 1$~~

(B) $y = x^2 + 2$

(C) ~~$y = x^3 - 2$~~

(D) ~~$y = \ln(x + 1)$~~

(E) ~~$y = 2e^x$~~

An initial value problem can have 0, 1, or multiple equilibrium solutions.

************Existence of a solution*************

************Uniqueness of solution*************

1.3:

ODE (ordinary differential equation): single independent variable

Ex: $\boxed{\frac{dy}{dt} = ay + b}$

PDE (partial differential equation): several independent variables

Ex: $\frac{\partial xy}{\partial x} = \frac{\partial xy}{\partial y}$

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order of differential eq'n: order of highest derivative

example of order n: $y^{(n)} = f(t, y, ..., y^{(n-1)})$ Ch2

Ch3 Ch4

1rst order 2nd order higher

Linear vs Non-linear

Linear: $a_0 y^{(n)} + ... + a_{n-1} y' + a_n y = g(t)$

where a_i's are functions of t

linear equations w/constant coeff

Note for this linear equation, the left hand side is a linear combination of the derivatives of y (denoted by $y^{(k)}, k = 0, ..., n$) where the coefficient of $y^{(k)}$ is a function of t (denoted $a_k(t)$).

Linear: $a_0(t)y^{(n)} + ... + a_{n-1}(t)y' + a_n(t)y = g(t)$ ← linear

Determine if linear or non-linear:

Ex: $ty'' - t^3 y' - 3y = sin(t)$ ← linear

Ex: $2y'' - 3y' - 3y^2 = 0$ ← not linear

Show that for some value of r, $y = e^{rt}$ is a soln to the 1^{rst} order linear homogeneous equation $2y' - 6y = 0$.

To show something is a solution, plug it in:

$y = e^{rt}$ implies $y' = re^{rt}$. Plug into $2y' - 6y = 0$:

$2re^{rt} + 6e^{rt} = 0$ implies $2r - 6 = 0$ implies $r = 3$

Thus $y = e^{3t}$ is a solution to $2y' - 6y = 0$.

Show $y = Ce^{3t}$ is a solution to $2y' - 6y = 0$.

$$2y' - 6y = 2(Ce^{3t})' - 6(Ce^{3t}) = 2C(e^{3t})' - 6C(e^{3t}) = C[2(e^{3t})' - 6(e^{3t})] = C(0) = 0.$$

Foreshadowing ch3 & 4

If $y(0) = 4$, then $4 = Ce^{3(0)}$ implies $C = 4$.

Thus by existence and uniqueness thm, $y = 4e^{3t}$ is the unique solution to IVP: $2y' + 6y = 0$, $y(0) = 4$.

CH 2: Solve $\frac{dy}{dt} = f(t, y)$ for special cases:

2.2: Separation of variables: $\int N(y)dy = \int P(t)dt$

2.1: First order linear eqn: $\frac{dy}{dt} + p(t)y = g(t)$

Ex 1: $t^2y' + 2ty = tsin(t)$ ← linear, not separable

Ex 2: $y' = ay + b$ ← linear, separable

Ex 3: $y' + 3t^2y = t^2$, $y(0) = 0$ ← linear, separable

Note: can use either section 2.1 method (integrating factor) or 2.2 method (separation of variables) to solve ex 2 and 3.

Ex 1: $t^2y' + 2ty = sin(t)$
(note, cannot use separation of variables).

product rule

$t^2y' + 2ty = sin(t)$

$(t^2y)' = sin(t)$ implies $\int (t^2y)'dt = \int sin(t)dt$

$(t^2y) = -cos(t) + C$ implies $y = -t^{-2}cos(t) + Ct^{-2}$

Ex. 2: Solve $\frac{dy}{dt} = ay + b$ by separating variables:

$\frac{dy}{ay+b} = dt \Rightarrow \int \frac{dy}{ay+b} = \int dt \Rightarrow \frac{ln|ay+b|}{a} = t + C$

$ln|ay + b| = at + C$ implies $e^{ln|ay+b|} = e^{at+C}$

$|ay + b| = e^Ce^{at}$ implies $ay + b = \pm(e^Ce^{at})$

$ay = Ce^{at} - b$ implies $y = Ce^{at} - \frac{b}{a}$

Gen ex: Solve $1y' + p(x)y = g(x)$

Let $F(x)$ be an anti-derivative of $p(x)$. Thus $p(x) = F'(x)$

$e^{F(x)}y' + [p(x)e^{F(x)}]y = g(x)e^{F(x)}$

$e^{F(x)}y' + [F'(x)e^{F(x)}]y = g(x)e^{F(x)}$

product rule

$[e^{F(x)}y]' = g(x)e^{F(x)}$

$e^{F(x)}y = \int g(x)e^{F(x)}dx$

$y = e^{-F(x)} \int g(x)e^{F(x)}dx$

Integrating factor $e^{\int p(x)dx}$

1rst order linear equ

$\Rightarrow$

$f : A \to B$ is 1:1 iff $f(x_1) = f(x_2)$ implies $x_1 = x_2$.

$$f(x_1) = f(x_2) \text{ implies } x_1 = x_2.$$

Hypothesis: $f(x_1) = f(x_2)$. Conclusion $x_1 = x_2$.

Hypothesis implies conclusion.
p implies q.
$$p \Rightarrow q.$$

Note a statement, $p \Rightarrow q$, is true if whenever the hypothesis p holds, then the conclusion q also holds.

To prove that a statement is true:
(1) Assume the hypothesis holds.
(2) Prove the conclusion holds.

Ex: To prove a function is 1:1:
(1) Assume $f(x_1) = f(x_2)$
(2) Do some algebra to prove $x_1 = x_2$.

$[p \Rightarrow q]$ is equivalent to $[\forall p, q$ holds$]$.

That is, for everything satisfying the hypothesis p, the conclusion q must hold.

A statement is false if the hypothesis holds, but the conclusion need not hold.

Hypothesis does not implies conclusion.
p does not imply q.
$$p \not\Rightarrow q.$$

That is there exists **a specific case** where the hypothesis holds, but the conclusion does not hold.

To prove that a statement is false: Look counter-example

Find an example where the hypothesis holds, but the conclusion does not hold.

Ex: To prove a function is not 1:1, find specific x_1, x_2 such that $f(x_1) = f(x_2)$, but $x_1 \neq x_2$.

Ex: $f : R \to R$, $f(x) = x^2$ is not 1:1 since $f(1) = 1^2 = 1 = (-1)^2 = f(-1)$, but $1 \neq -1$

$\sim [p \Rightarrow q]$ is equivalent to $\sim [\forall p, q$ holds$]$.

Thus if $p \Rightarrow q$ is false,
then it is not true that $[\forall p, q$ holds$]$.
That is, $\exists p$ such that q does not hold.

$p \Rightarrow q$ is false

$\forall$ = for all

$\exists$ = there exists

$f : A \to B$ is 1:1 iff $f(x_1) = f(x_2)$ implies $x_1 = x_2$.

$f : A \to B$ is 1:1 iff $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$.

$f : A \to B$ is 1:1 iff for all $x_1 \neq x_2$, $f(x_1) \neq f(x_2)$.

$f : A \to B$ is NOT 1:1 iff there exists $x_1 \neq x_2$ such that $f(x_1) = f(x_2)$.

Determine if the following functions are 1:1. Prove it.

~~1.) $f : R \to R$, $f(x) = x^2$~~

$f(-1) = f(1)$ but $-1 \neq 1$ $\Rightarrow$ not 1:1

Goal: $X_1 = X_2$

2.) $f : [0, \infty) \to R$, $f(x) = x^2$

Suppose $f(x_1) = f(x_2) \Rightarrow X_1^2 = X_2^2 \Rightarrow X_1 = \pm\sqrt{X_2^2} = \pm|X_2|$

$X_1, X_2 \in [0, \infty) \Rightarrow X_1 \nleftarrow X_2 \geq 0$

$X_1 = \pm|X_2| \Rightarrow X_1 = X_2$

3.) $f : [0, \infty) \to [0, \infty)$, $f(x) = x^2$

4.) $f : R \to R$, $f(x) = x^3$

5.) $f : R \to R$, $f(x) = 2$

6.) $f : R \to R$, $f(x) = 8x + 2$

7.) $f : R \to R$, $f(x) = x^2 + 3x$

8.) $f : R \to R$, $f(x) = e^x$

9.) $f : R \to R$, $f(x) = x^4 + x^2$

10.) $f : R \to R$, $f(x) = sin(x)$

there does not exist

$f : A \to B$ is onto iff $f(A) = B$.

$f : A \to B$ is onto iff $b \in B$ implies there exists an $a \in A$ such that $f(a) = b$.

$f : A \to B$ is onto iff for all $b \in B$, there exists an $a \in A$ such that $f(a) = b$.

$f : A \to B$ is NOT onto iff there exists $b \in B$ s. t. there does not exist an $a \in A$ s. t. $f(a) = b$.

Determine if the following functions are onto. If a function is not onto, prove it.

1.) $f : R \to R$, $f(x) = x^2$

$-5 \in \mathbb{R}$ but $\nexists x$ st $f(x) = x^2 = -5$ not onto

2.) $f : [0, \infty) \to R$, $f(x) = x^2$

3.) $f : [0, \infty) \to [0, \infty)$, $f(x) = x^2$

onto

4.) $f : R \to R$, $f(x) = x^3$

5.) $f : R \to R$, $f(x) = 2$

6.) $f : R \to R$, $f(x) = 8x + 2$

7.) $f : R \to R$, $f(x) = x^2 + 3x$

8.) $f : R \to R$, $f(x) = e^x$

9.) $f : R \to R$, $f(x) = x^4 + x^2$

10.) $f : R \to R$, $f(x) = sin(x)$