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$r^5 = 32 \Rightarrow r = (32)^{1/5}(1)^{1/5} = 2(e^{i0})^{1/5}$

$\Rightarrow r = 2(e^{i(0+2\pi k)})^{1/5} = 2e^{2\pi i k/5}$ = continue simplifying using Euler's formula

for $k = 0, 1, 2, 3, 4$

$r^3 = -27 \Rightarrow r = (27)^{1/3}(-1)^{1/3}$

$\Rightarrow r = 3(e^{i\pi})^{1/3} = 3\left(e^{i(\pi+2\pi k)/3}\right)$ $k = 0, 1, 2$

= continue simplifying using Euler's formula: $e^{i\theta} = \cos\theta + i\sin\theta$

Note we used: $1 = e^{i(0+2\pi k)}$

and $-1 = e^{i(\pi+2\pi k)}$

Thm 4.1.1 example

$$t\, y^{iv} - \frac{3y'''}{(t-2)} + \sqrt{t}\, y = \frac{\ln|t^4|}{t^2-9}$$

$$\Rightarrow 1 y^{iv} - \frac{3y'''}{t(t-2)} + \frac{\sqrt{t}\, y}{t} = \frac{\ln|t^4|}{t(t-3)(t+3)}$$

p_i, g cont $\Leftrightarrow t \neq 0, 2, 3, -3,$ & $t \geq 0$ & $t \neq 4$

$\Leftrightarrow t > 0$ & $t \neq 2, 3, 4$

If $y(5) = y_0 \Rightarrow$ solution valid on $(4, \infty)$

If $y(1) = y_0 \Rightarrow$ " " " $(0, 1)$

Solve $y''' + y'' + 3y' + 10y = 0$

Guess $y = e^{rt}$

Solving polynomial equations:

Example: $r^3 + r^2 + 3r + 10 = 0$

Plug in $r = \pm 1, \pm 2, \pm 5, \pm 10$ to see if any of these are solns:

$(\pm 1)^3 + (\pm 1)^2 + 3(\pm 1) + 10 \neq 0$

$(\pm 2)^3 + (\pm 2)^2 + 3(\pm 2) + 10 \ ? =?0$

$(-2)^3 + (-2)^2 + 3(-2) + 10 = -8 + 4 - 6 + 10 = 0$ ✓

Thus $(r - (-2))$ is a factor of $r^3 + r^2 + 3r + 10$

Hence $r^3 + \underline{1}r^2 + 3r + \boxed{10} = (r + \boxed{2})(r^2 + \underline{-1}\, r + \boxed{5})$

$5 = \frac{10}{2}$

$1r^2 = 2r^2 + -1r^2$

To find the coefficient of r in the above, you can do so by (1) long division, (2) inspection, (3) using variable x

$r^3 + r^2 + 3r + 10 = (r+2)(r^2 + \underline{\ x\ }\, r + 5)$

$(r+2)(r^2 + \underline{\ x\ }\, r + 5) = r^3 + (2 + x)r^2 + (2x + 5)r + 10$

$r^3 \quad + \quad r^2 \quad + \quad 3r \quad + \quad 10$

Thus $2 + x = 1$ and hence $x = -1$

Check: $2x + 5 = 2(-1) + 5 = 3$

Hence $r^3 + r^2 + 3r + 10 = (r+2)(r^2 - r + 5) = 0$

Thus $r = -2, \frac{1 \pm \sqrt{1-20}}{2}$.

Thus $r = -2, \frac{1 \pm i\sqrt{19}}{2}$.

Guessed $y = e^{rt}$

$y = c_1 (e^{-2t}) + c_2 \left(e^{\frac{1}{2}t} \cos\left(\frac{\sqrt{19}}{2}t\right)\right) + c_3 \left(e^{\frac{1}{2}t} \sin\left(\frac{\sqrt{19}}{2}t\right)\right)$

$y^{(n)} + b = 0$

In special cases, you can use the unit circle.

Ex: $r^4 + 1 = 0$ implies

$r = (-1)^{\frac{1}{4}} = (-1 + 0i)^{\frac{1}{4}} = (e^{i\pi})^{\frac{1}{4}} = (e^{i(\pi + 2\pi k)})^{\frac{1}{4}}$

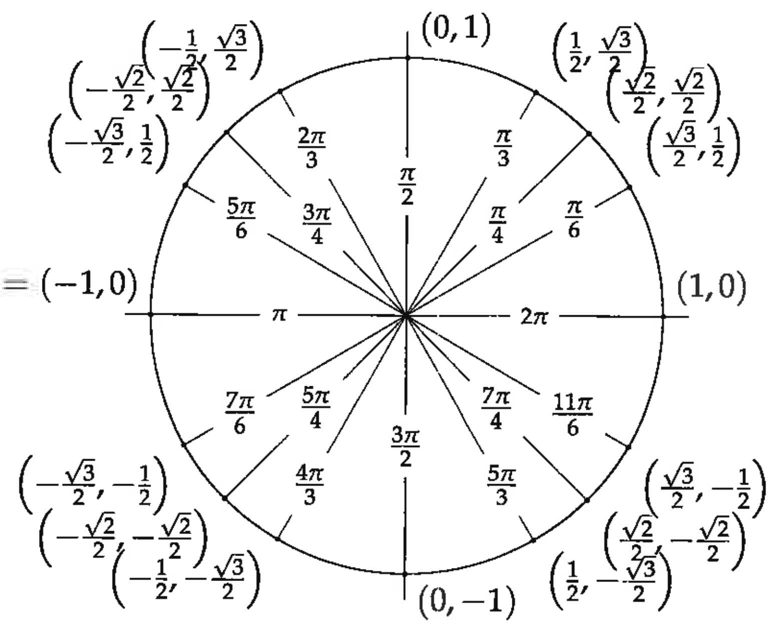
$k = 0: \quad e^{\frac{i\pi}{4}} = cos(\frac{i\pi}{4}) + isin(\frac{i\pi}{4}) = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$

$k = 1: \quad e^{\frac{3i\pi}{4}} = cos(\frac{3i\pi}{4}) + isin(\frac{3i\pi}{4}) = -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$

$k = 2: \quad e^{\frac{5i\pi}{4}} = cos(\frac{5i\pi}{4}) + isin(\frac{5i\pi}{4}) = -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$

$k = 3: \quad e^{\frac{7i\pi}{4}} = cos(\frac{7i\pi}{4}) + isin(\frac{7i\pi}{4}) = \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$

$-1 + 0i = (-1, 0)$



nth order LINEAR differential equation:

Thm 2.4.1: If p and g are continuous on (a, b) and the point $t_0 \in (a, b)$, then there exists a unique function $y = \phi(t)$ defined on (a, b) that satisfies the following initial value problem:

$$1\, y' + p(t)y = g(t), \quad y(t_0) = y_0.$$

Thm 3.2.1: If $p : (a, b) \to R$, $q : (a, b) \to R$, and $g : (a, b) \to R$ are continuous and $a < t_0 < b$, then there exists a unique function $y = \phi(t)$, $\phi : (a, b) \to R$ that satisfies the initial value problem

$$1\, y'' + p(t)y' + q(t)y = g(t),$$
$$y(t_0) = y_0, \quad y'(t_0) = y_1$$

Theorem 4.1.1: If $p_i : (a, b) \to R$, $i = 1, ..., n$ and $g : (a, b) \to R$ are continuous and $a < t_0 < b$, then there exists a unique function $y = \phi(t)$, $\phi : (a, b) \to R$ that satisfies the initial value problem

$$1\, y^{(n)} + p_1(t)y^{(n-1)} + ... + p_{n-1}(t)y' + p_n(t)y = g(t),$$
$$y(t_0) = y_0, \quad y'(t_0) = y_1, \; ..., \quad y^{(n-1)}(t_0) = y_{n-1}$$

Proof: We proved the case $n = 1$ using an integrating factor. When $n > 1$, see more advanced textbook.

Example 4 from ch 2: $(t^2-1)y' + \frac{(t+1)y}{t-4} = ln|t|, \; y(3) = 6$

This equation is linear, so we know that it has a unique solution as long as p and g are continuous.

$$(t^2-1)y' + \frac{(t+1)y}{t-4} = ln|t| \Rightarrow y' + \frac{(t+1)}{(t-4)(t^2-1)}y = \frac{ln|t|}{t^2-1}$$

Note $p(t) = \frac{(t+1)}{(t-4)(t^2-1)} = \frac{\cancel{(t+1)}}{(t-4)\cancel{(t+1)}(t-1)} = \frac{1}{(t-4)(t-1)}$ is continuous for all $t \neq 1, 4$

Note $g(t) = \frac{ln|t|}{t^2-1} = \frac{ln|t|}{(t+1)(t-1)}$ is continuous for all $t \neq -1, 0, 1$

Thus $ty' - y = 1, \; y(t_0) = y_0$ has a unique solution as long as $t_0 \neq -1, 0, 1, 4$.

Since for IVP, $(t^2-1)y' + \frac{(t+1)y}{t-4} = ln|t|, \; y(3) = 6$, $t_0 = 3$, this IVP has a unique solution which by Thm 4.1.1 is valid on the interval $(1, 4)$.

NOTE: Theorem 4.1.1 is VERY useful in the real world. Suppose you can't solve the linear differential equation directly. You may be able to instead approximate the solution – see for example ch 5 series solution (guess $y = \sum a_n x^n$), which we won't cover in this class or MATH:3800 Elementary Numerical Analysis.

But your approximation is not of much use unless you know where your approximation is valid.

To solve $ay'' + by' + cy = g(t)$

1.) Easily solve homogeneous DE: $ay'' + by' + cy = 0$

$y = e^{rt} \Rightarrow ar^2 + br + c = 0 \Rightarrow y = c_1\phi_1 + c_2\phi_2$ for homogeneous solution (see sections 3.1, 3.3, 3.4, 4.1).

2.) More work: Find one solution to $ay'' + by' + cy = g(t)$ (see sections $3.5 = 4.3$, $3.6 = 4.4$)

If $y = \psi(t)$ is a soln to the nonhomogeneous DE, then general soln to $ay'' + by' + cy = g(t)$ is

$$y = c_1\phi_1 + c_2\phi_2 + \psi$$

Check: $a\phi_1'' + b\phi_1' + c\phi_1 = 0$
$a\phi_2'' + b\phi_2' + c\phi_2 = 0$
$a\psi'' + b\psi' + c\psi = g(t)$

Step 3: If IVP plug in initial values to find c_i's

Note you can break step 2 into simpler parts. For example:

To solve $ay'' + by' + cy = g_1(t) + g_2(t)$

1.) Solve $ay'' + by' + cy = 0 \quad \Rightarrow \quad y = c_1\phi_1 + c_2\phi_2$ for homogeneous solution.

2a.) Solve $ay'' + by' + cy = g_1(t) \quad \Rightarrow \quad y = \psi_1$

2b.) Solve $ay'' + by' + cy = g_2(t) \quad \Rightarrow \quad y = \psi_2$

General solution to $ay'' + by' + cy = g_1(t) + g_2(t)$ is

$$y = c_1\phi_1 + c_2\phi_2 + \psi_1 + \psi_2$$

$0 + 0 + g_1 + g_2$

LAST STEP

When does the following IVP have unique sol'n:

IVP: $ay'' + by' + cy = g(t)$, $y(t_0) = y_0$, $y'(t_0) = y_1$.

Suppose $y = c_1\phi_1(t) + c_2\phi_2(t) + \psi(t)$ is a solution to $ay'' + by' + cy = g(t)$. Then $y' = c_1\phi_1'(t) + c_2\phi_2'(t) + \psi'(t)$

$y(t_0) = y_0$: $\quad y_0 = c_1\phi_1(t_0) + c_2\phi_2(t_0) + \psi(t_0)$

$y'(t_0) = y_1$: $\quad y_1 = c_1\phi_1'(t_0) + c_2\phi_2'(t_0) + \psi'(t_0)$

To find IVP solution, need to solve above system of two equations for the unknowns c_1 and c_2.

Note the IVP has a unique solution if and only if the above system of two equations has a unique solution for c_1 and c_2.

Note that in these equations c_1 and c_2 are the unknowns.

Let $b_0 = y_0 - \psi(t_0)$ and $b_1 = y_1 - \psi'(t_0)$

We can translate this linear system of equations into matrix form:

$$\begin{array}{l} c_1\phi_1(t_0) + c_2\phi_2(t_0) = b_0 \\ c_1\phi_1'(t_0) + c_2\phi_2'(t_0) = b_1 \end{array} \Rightarrow \begin{bmatrix} \phi_1(t_0) & \phi_2(t_0) \\ \phi_1'(t_0) & \phi_2'(t_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}$$

Note this equation has a unique solution if and only if

$$det \begin{bmatrix} \phi_1(t_0) & \phi_2(t_0) \\ \phi_1'(t_0) & \phi_2'(t_0) \end{bmatrix} = \begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix} = \phi_1\phi_2' - \phi_1'\phi_2 \neq 0$$

Definition: The Wronskian of two differential functions, ϕ_1 and ϕ_2 is

$$W(\phi_1, \phi_2)(t) = (\phi_1\phi_2' - \phi_1'\phi_2)(t) = \begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix}(t)$$

compare to coeff matrix when solving for c's using initial values

Examples:

1.) $$W(cos(t),\, sin(t)) = \begin{vmatrix} cos(t) & sin(t) \\ -sin(t) & cos(t) \end{vmatrix}$$

$$= cos^2(t) + sin^2(t) = 1 > 0.$$

2.) $$W(e^{dt}cos(nt),\, e^{dt}sin(nt)) =$$

$$\begin{vmatrix} e^{dt}cos(nt) & e^{dt}sin(nt) \\ de^{dt}cos(nt) - ne^{dt}sin(nt) & de^{dt}sin(nt) + ne^{dt}cos(nt) \end{vmatrix}$$

$$=e^{dt}cos(nt)(de^{dt}sin(nt)+ne^{dt}cos(nt))-e^{dt}sin(nt)(de^{dt}cos(nt)-ne^{dt}sin(n$$

$$=e^{2dt}[cos(nt)(dsin(nt)+ncos(nt))-sin(nt)(dcos(nt)-nsin(nt))]$$

$$=e^{2dt}[dcos(nt)sin(nt)+ncos^2(nt)-dsin(nt)cos(nt)+nsin^2(nt)])$$

$$= e^{2dt}[ncos^2(nt) + nsin^2(nt)]$$

$$= ne^{2dt}[cos^2(nt) + sin^2(nt)] = ne^{2dt} > 0 \text{ for all } t.$$

4.1: General Theory of nth Order Linear Eqns

When does the following IVP have a unique soln:

IVP: $y^{(n)} + p_1(t)y^{(n-1)} + ... + p_{n-1}(t)y' + p_n(t)y = g(t)$,

$$y(t_0) = y_0,\ y'(t_0) = y_1, ..., y^{(n-1)}(t_0) = y_{n-1}.$$

initial values

Suppose $y = c_1\phi_1(t) + c_2\phi_2(t) + ... + c_n\phi_n(t) + \psi(t)$ is the general solution to DE. Then

$y(t_0) = y_0$:

$$y_0 = c_1\phi_1(t_0) + c_2\phi_2(t_0) + ... + c_n\phi_n(t_0) + \psi(t_0)$$

$y'(t_0) = y_1$:

$$y_1 = c_1\phi_1'(t_0) + c_2\phi_2'(t_0) + ... + c_n\phi_n'(t_0) + \psi'(t_0)$$

$$\vdots$$

$y^{(n-1)}(t_0) = y_{n-1}$:

$$y_{n-1} = c_1\phi_1^{(n-1)}(t_0) + c_2\phi_2^{(n-1)}(t_0) + ... + c_n\phi_n^{(n-1)}(t_0) + \psi^{(n-1)}(t_0)$$

To find IVP solution, need to solve above system of equations for the unknowns c_i, $i = 1, ..., n$.

Note the IVP has a unique solution if and only if the above system of equations has a unique solution for the c_i's.

Let $b_k = y_k - \psi^{(k)}(t_0)$. Note that in these equations the c_i are the unknowns

Translating this linear system of eqns into matrix form:

$$\begin{bmatrix} \phi_1(t_0) & \phi_2(t_0) & \dots & \phi_n(t_0) \\ \phi_1'(t_0) & \phi_2'(t_0) & \dots & \phi_n'(t_0) \\ & \vdots & & \\ \phi_1^{(n-1)}(t_0) & \phi_2^{(n-1)}(t_0) & \dots & \phi_n^{(n-1)}(t_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{bmatrix}$$

this is coef matrix used to find c_i's

Note this equation has a unique solution if and only if

$$det \begin{bmatrix} \phi_1(t_0) & \phi_2(t_0) & \dots & \phi_n(t_0) \\ \phi_1'(t_0) & \phi_2'(t_0) & \dots & \phi_n'(t_0) \\ & \vdots & & \\ \phi_1^{(n-1)}(t_0) & \phi_2^{(n-1)}(t_0) & \dots & \phi_n^{(n-1)}(t_0) \end{bmatrix} \neq 0$$

Before taking deter

Evaluated at t_0

Defn: The Wronskian of the functions, $\phi_1, \phi_2, \dots, \phi_n$ is

$$W(\phi_1, \phi_2, \dots, \phi_n)(t) = det \begin{bmatrix} \phi_1(t) & \phi_2(t) & \dots & \phi_n(t) \\ \phi_1'(t) & \phi_2'(t) & \dots & \phi_n'(t) \\ & \vdots & & \\ \phi_1^{(n-1)}(t) & \phi_2^{(n-1)}(t) & \dots & \phi_n^{(n-1)}(t) \end{bmatrix}$$

Note: $\{\phi_1, \phi_2, \dots, \phi_n\}$ is a linearly independent set of fns if $W(\phi_1, \phi_2, \dots, \phi_n)(t_0) \neq 0$ for some t_0

In other words if ϕ_i are homogeneous solutions to an nth order linear DE,

$$y^{(n)} + p_1(t)y^{(n-1)} + ... + p_{n-1}(t)y' + p_n(t)y = 0$$

and $W(\phi_1, \phi_2, ..., \phi_n)(t_0) \neq 0$ for some t_0.

iff $\{\phi_1, \phi_2, ..., \phi_n\}$ is a basis for the solution set of this homogeneous equation.

In other words any homogeneous solution can be written as a linear combination of these basis elements:

$$y = c_1\phi_1 + ... + c_n\phi_n$$

Moreover, the general soln to the non-homogeneous eqn

$$y^{(n)} + p_1(t)y^{(n-1)} + ... + p_{n-1}(t)y' + p_n(t)y = g(t)$$

is just the translated version of the general homogeneous solution:

$$y = c_1\phi_1 + ... + c_n\phi_n + \psi$$

where ψ is a non-homogeneous solution.

Abel's theorem: if ϕ_i are homogeneous solutions to an nth order linear DE,

1 $$y^{(n)} + p_1(t)y^{(n-1)} + ... + p_{n-1}(t)y' + p_n(t)y = 0$$

then $W(\phi_1, \phi_2, ..., \phi_n)(t) = ce^{-\int p_1(t)dt}$ for some constant c

FYI

$\phi_1, \dots, \phi_n$ are linearly independent

iff

$c_1\phi_1(t) + \dots + c_n\phi_n(t) = 0$ has a unique solution (that works for all t).

$\boxed{\text{iff}}$

the following system of equations has a unique solution

$$\begin{aligned}
c_1\phi_1(t) + c_2\phi_2(t) + \dots + c_n\phi_n(t) &= 0 \\
c_1\phi_1'(t) + c_2\phi_2'(t) + \dots + c_n\phi_n'(t) &= 0 \\
&\vdots \\
c_1\phi_1^{(n-1)}(t) + c_2\phi_2^{(n-1)}(t) + \dots + c_n\phi_n^{(n-1)}(t) &= 0
\end{aligned}$$

take derivative

iff the following system of equations has a unique solution

$$\begin{bmatrix} \phi_1(t) & \phi_2(t) & \cdots & \phi_n(t) \\ \phi_1'(t) & \phi_2'(t) & \cdots & \phi_n'(t) \\ & \vdots & & \\ \phi_1^{(n-1)}(t) & \phi_2^{(n-1)}(t) & \cdots & \phi_n^{(n-1)}(t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Note this equation has a unique solution if and only if for some t_0

$$det \begin{bmatrix} \phi_1(t_0) & \phi_2(t_0) & \cdots & \phi_n(t_0) \\ \phi_1'(t_0) & \phi_2'(t_0) & \cdots & \phi_n'(t_0) \\ & \vdots & & \\ \phi_1^{(n-1)}(t_0) & \phi_2^{(n-1)}(t_0) & \cdots & \phi_n^{(n-1)}(t_0) \end{bmatrix} \neq 0$$

iff $\quad W(\phi_1, \phi_2, ..., \phi_n)(t_0) \neq 0,$

Example: Determine if $\{1+2t, 5+4t^2, 6-8t+8t^2\}$ are linearly independent:

Method 1:

$$\boxed{\text{Solve } c_1(1+2t) + c_2(5+4t^2) + c_3(6-8t+8t^2) = 0}$$

Or equivalently,
solve $c_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 5 \\ 0 \\ 4 \end{bmatrix} + c_3 \begin{bmatrix} 6 \\ -8 \\ 8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Or equivalently, solve $\begin{bmatrix} 1 & 5 & 6 \\ 2 & 0 & -8 \\ 0 & 4 & 8 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Method 2: Check the Wronskian

$$det \begin{bmatrix} 1+2t & 5+4t^2 & 6-8t+8t^2 \\ 2 & 8t & -8+16t \\ 0 & 8 & 16 \end{bmatrix}$$

if $= 0 \Rightarrow$ lin dep
$\neq 0 \Rightarrow$ lin indep