3.7 Mechanical Vibrations:

$$\boxed{mu''(t) + \gamma u'(t) + ku(t) = F_{external},} \quad m, \gamma, k \geq 0$$

$$mg - kL = 0, \qquad F_{damping}(t) = -\gamma u'(t)$$

viscous

$m=$ mass, $k > 0$ $\gamma \geq 0$

$k =$ spring force proportionality constant,

$\gamma =$ damping force proportionality constant

$g = 9.8$ m/sec^2 or 32 ft/sec^2. Weight $= mg$.

$m > 0$

Electrical Vibrations:

$$L\frac{dI(t)}{dt} + RI(t) + \frac{1}{C}Q(t) = E(t), \quad L, R, C \geq 0 \text{ and } I = \frac{dQ}{dt}$$

$$LQ''(t) + RQ'(t) + \frac{1}{C}Q(t) = E(t)$$

CIRCUITS

$L =$ inductance (henrys),

$R =$ resistance (ohms)

$C =$ capacitance (farads)

$Q(t) =$ charge at time t (coulombs)

$I(t) =$ current at time t (amperes)

$E(t) =$ impressed voltage (volts).

1 volt = 1 ohm · 1 ampere = 1 coulomb / 1 farad = 1 henry · 1 amperes/ 1 second

$L = 4$

Suppose a mass weighing 64 lbs stretches a spring 4 ft. If there is no damping and the spring is stretched an additional foot and set in motion with an upward velocity of $\sqrt{8}$ ft/sec, find the equation of motion of the mass.

$\gamma = 0$

$Weight = mg$: $m = \frac{weight}{g} = \frac{64}{32} = 2$

$mg - kL = 0$ implies $k = \frac{mg}{L} = \frac{64}{4} = 16$

$mu''(t) + \gamma u'(t) + ku(t) = F_{external}$

[$\gamma^2 - 4km < 0$: $u(t) = e^{-\frac{\gamma t}{2m}}(Acos\mu t + Bsin\mu t)$
Hence $u(t) = Acos\mu t + Bsin\mu t$ since $\gamma = 0$].

$\boxed{2u''(t) + 16u(t) = 0}$ up

positive since pulled down

$u''(t) + 8u(t) = 0$, $\boxed{u(0) = 1,\ u'(0) = -\sqrt{8}}$

upward

$r^2 + 8 = 0 \rightarrow r = \pm\sqrt{-8} = \pm i\sqrt{8} = 0 \pm i\sqrt{8}$

$u(t) = c_1 e^{it\sqrt{8}} + c_2 e^{-it\sqrt{8}}$ ← not ~~ampl~~ simplified

$u(t) = Acos\sqrt{8}t + Bsin\sqrt{8}t$ $\boxed{\omega_0 = \text{frequency}}$

For word problems simplify further $R\cos(\omega_0 t - \delta)$

initial values

$u(0) = 1$: $1 = Acos(0) + Bsin(0) = A$

$u'(t) = -\sqrt{8}Asin\sqrt{8}t + \sqrt{8}Bcos\sqrt{8}t$

$u'(0) = -\sqrt{8}$: $-\sqrt{8} = -\sqrt{8}Asin(0) + \sqrt{8}Bcos(0)$

$B = -1$ Thus $u(t) = cos\sqrt{8}t - sin\sqrt{8}t$

$u(t) = 1cos(t\sqrt{8}) - 1sin(t\sqrt{8}) = Rcos(t\sqrt{8} - \delta)$

$$u = \sqrt{2}cos(t\sqrt{8} - \tfrac{7\pi}{4}) = \sqrt{2}cos(t\sqrt{8} + \tfrac{\pi}{4}) = u$$

$A = 1 = Rcos(\delta)$ and $B = -1 = Rsin(\delta)$

Thus $R = \sqrt{A^2 + B^2} = \sqrt{(1)^2 + (-1)^2} = \sqrt{2}$

$tan(\delta) = \frac{Rsin(\delta)}{Rcos(\delta)} = \frac{B}{A} = \frac{-1}{1}$. Thus $\delta = -\frac{\pi}{4} = \frac{7\pi}{4}$

Trig background:

$$cos(y \mp x) = cos(x \mp y) = cos(x)cos(y) \pm sin(x)sin(y)$$

Let $A = Rcos(\delta)$, $B = Rsin(\delta)$ in

$$\begin{aligned} &Acos(\omega_0 t) + Bsin(\omega_0 t) \\ &\quad = Rcos(\delta)cos(\omega_0 t) + Rsin(\delta)sin(\omega_0 t) \\ &\quad = Rcos(\omega_0 t - \delta) \end{aligned}$$

want

Amplitude $= R$ calculate

frequency $= \omega_0$ (measured in radians per unit time).

period $= \frac{2\pi}{\omega_0}$ phase (displacement) $= \delta$

calculate

$A = Rcos(\delta)$, $B = Rsin(\delta)$ implies

$$\begin{aligned} A^2 + B^2 &= R^2cos^2(\delta) + R^2sin^2(\delta) \\ &= R^2(cos^2(\delta) + sin^2(\delta)) = R^2 \end{aligned}$$

Mechanical Vibrations:

$$mu''(t) + \gamma u'(t) + ku(t) = F_{external}, \quad m, \gamma, k \geq 0$$

$$\text{IVP: } u(t_0) = u_0, \quad , u'(t_0) = u_1$$

NOTE: Positive direction points DOWN.

m= mass,

k = spring force proportionality constant,

γ = damping force proportionality constant

g = 9.8 m/sec^2 or 32 ft/sec^2.

Weight $= mg \qquad mg - kL = 0, \qquad F_{damping}(t) = -\gamma u'(t)$

3.8 A mass of 3kg stretches a spring 4.9 m. If the mass is acted upon by an external force of $40e^{\frac{-t}{3}}$ N in a medium that imparts a viscous force of 10 N when the speed of the mass is 5 m/sec. If the mass is pulled down 1 m and set in motion with an upward velocity of 8 m/sec, describe the motion of the mass.

damping force

$m = 3$

$|F_{damping}(t)| = |\gamma u'(t)| \Rightarrow 10 = \gamma(5)$. Thus $\gamma = 2$

$mg - kL = 0$ implies $= \frac{mg}{L} = \frac{3(9.8)}{4.9} = 6$

IVP: $3u''+2u'+6u = 40e^{\frac{-t}{3}}, \quad u(0) = 1, \quad u'(0) = -8$

IVP: $3u''+2u'+6u = 40e^{\frac{-t}{3}}, \quad u(0) = 1, \quad u'(0) = -8$

Step 1: Solve homogeneous eqn: $3u'' + 2u' + 6u = 0$

$u = e^{rt}$ implies $3r^2 + 2r + 6 = 0$ implies

$$r = \frac{-2 \pm \sqrt{4 - 4(3)(6)}}{2(3)} = \frac{-2}{2(3)} \pm \frac{\sqrt{4}\sqrt{1-(3)(6)}}{2(3)} = -\frac{1}{3} \pm i\frac{\sqrt{17}}{3}$$

Thus general homogeneous solution is

$$u(t) = c_1 e^{-\frac{t}{3}} cos(\frac{\sqrt{17}}{3}t) + c_2 e^{-\frac{t}{3}} sin(\frac{\sqrt{17}}{3}t)$$

Step 2: Find a non-homogeneous soln

Section 3.6: First find Wronskian,

$$W(e^{-\frac{t}{3}} cos(\frac{\sqrt{17}}{3}t), e^{-\frac{t}{3}} sin(\frac{\sqrt{17}}{3}t)).$$

Too much work. Thus,

Section 3.5: Guess $u(t) = Ae^{\frac{-t}{3}}$.

Then $u'(t) = -\frac{A}{3}e^{\frac{-t}{3}}$ and $u''(t) = \frac{A}{9}e^{\frac{-t}{3}}$

$3(\frac{A}{9}e^{\frac{-t}{3}}) + 2(-\frac{A}{3}e^{\frac{-t}{3}}) + 6Ae^{\frac{-t}{3}} = 40e^{\frac{-t}{3}}$

$\frac{A}{3} - \frac{2A}{3} + 6A = 40$ implies $A - 2A + 18A = 17A = 120$.

Thus $A = \frac{120}{17}$ and

hence $u(t) = \frac{120}{17}e^{\frac{-t}{3}}$ is a non-homogeneous soln.