Find the solution to the initial value problem:

$$y'' - 6y' + 9y = 8e^{3t} + 27t, \quad y(0) = 5, \quad y'(0) = 2.$$

Step 1: Solve Homogeneous equation $y'' - 6y' + 9y = 0$

Let $y = e^{rt}$. Then $r^2 - 6r + 9 = 0$. Thus $(r-3)^2 = 0$ and $r = 3$.

Thus general homogeneous solution is $y = c_1e^{3t} + c_2te^{3t}$.

Step 2a: Solve Non-homogeneous equation $y'' - 6y' + 9y = 8e^{3t}$

Since $y = e^{3t}$ and $y = te^{3t}$ are homogeneous solutions, multiples of these cannot be solutions to the non-homogeneouse equation. Thus we will try multiplying by another t and try $y = At^2e^{3t}$.

$y = At^2e^{3t}$ implies $y' = 2Ate^{3t} + 3At^2e^{3t}$ and $y'' = 2Ae^{3t} + 6Ate^{3t} + 6Ate^{3t} + 9At^2e^{3t}$

$$= 2Ae^{3t} + 12Ate^{3t} + 9At^2e^{3t}$$

Plugging into $y'' - 6y' + 9y = 8e^{3t}$ and solve for A:

$$2Ae^{3t} + 12Ate^{3t} + 9At^2e^{3t} - 6(2Ate^{3t} + 3At^2e^{3t}) + 9(At^2e^{3t}) = 8e^{3t}$$

$$2Ae^{3t} + 12Ate^{3t} + 9At^2e^{3t} - 12Ate^{3t} - 18At^2e^{3t} + 9At^2e^{3t} = 8e^{3t}$$

$$2Ae^{3t} + (12-12)Ate^{3t} + (9-18+9)At^2e^{3t} = 8e^{3t}$$

$2Ae^{3t} = 8e^{3t}$ implies $2A = 8$ and thus $A = 4$.

Thus $y = 4t^2e^{3t}$ is anon-homogeneous solution to $y'' - 6y' + 9y = 8e^{3t}$.

Thus general non-homogeneous solution to $y'' - 6y' + 9y = 8e^{3t}$ is

$$y = c_1e^{3t} + c_2te^{3t} + 4t^2e^{3t}.$$

Step 2b: Solve Non-homogeneous equation $y'' - 6y' + 9y = 27t$

Guess $y = At + B$. Then $y' = A$ and $y'' = 0$.

Plugging into $y'' - 6y' + 9y = 27t$ and solve for A and B:

$$0 - 6A + 9(At + B) = 27t$$

$9At + 9B - 6A = 27t + 0$. Thus $9A = 27$ and $9B - 6A = 0$.

Hence $A = 3$ and $9B = 6A = 6(3)$ and thus $B = 2$.

Thus $y = 3t + 2$ is a non-homogeneous solution to $y'' - 6y' + 9y = 27t$.

Thus general non-homogeneous solution to $y'' - 6y' + 9y = 27t$ is

$$y = c_1e^{3t} + c_2te^{3t} + 3t + 2.$$

Hence general non-homogeneous solution to $y'' - 6y' + 9y = 8e^{3t} + 27t$ **is**

$$y = c_1e^{3t} + c_2te^{3t} + 4t^2e^{3t} + 3t + 2.$$

or equivalently,

$$y = e^{3t}(4t^2 + c_2t + c_1) + 3t + 2.$$

Step 3: Use initial values to solve for c_1 and c_2:

$y = e^{3t}(4t^2 + c_2t + c_1) + 3t + 2$ implies $y' = 3e^{3t}(4t^2 + c_2t + c_1) + e^{3t}(8t + c_2) + 3$

$y(0) = 5$: $5 = e^0(4(0)^2 + c_2(0) + c_1) + 3(0) + 2$

$5 = c_1 + 2$ implies $c_1 = 3$

$y'(0) = 2$: $2 = 3e^0(4(0)^2 + c_2(0) + c_1) + e^0(8(0) + c_2) + 3$

$2 = 3c_1 + c_2 + 3$ implies $c_2 = 2 - 3 - 3c_1 = 2 - 3 - 9 = -10$

Thus solution to IVP is $y = e^{3t}(4t^2 - 10t + 3) + 3t + 2$.

Quiz 3 Oct. 14, 2016

1.) Suppose $y = c_1e^{3t} + c_2te^{3t} + 4t^2e^{3t}$ is a solution to $y'' - 6y' + 9y = 8e^{3t}$. Find the solution to the initial value problem:

$$y'' - 6y' + 9y = 8e^{3t} + 27t, \quad y(0) = 5, \quad y'(0) = 2.$$

Note: Solving this IVP is a 4 part problem, but I have already done the first two parts for you.

ANSWER: Since $y = c_1e^{3t} + c_2te^{3t} + 4t^2e^{3t}$ is a solution to $y'' - 6y' + 9y = 8e^{3t}$, we know $y = c_1e^{3t} + c_2te^{3t}$ is the general solution to the homogeneous equation is a solution to $y'' - 6y' + 9y = 0$ and $y = 4t^2e^{3t}$ is a solution to $y'' - 6y' + 9y = 8e^{3t}$.

Thus parts 1 and 2a are already completed. Repeating the remaining 2 parts:

Step 2b: Solve Non-homogeneous equation $y'' - 6y' + 9y = 27t$: Guess $y = At + B$. Then $y' = A$ and $y'' = 0$.

Plugging into $y'' - 6y' + 9y = 27t$ and solve for A and B: $0 - 6A + 9(At + B) = 27t$

$9At + 9B - 6A = 27t + 0$. Thus $9A = 27$ and $9B - 6A = 0$.

Hence $A = 3$ and $9B = 6A = 6(3)$ and thus $B = 2$.

Thus $y = 3t + 2$ is a non-homogeneous solution to $y'' - 6y' + 9y = 27t$.

Thus general non-homogeneous solution to $y'' - 6y' + 9y = 27t$ is $y = c_1e^{3t} + c_2te^{3t} + 3t + 2$.

Thus general non-homog. soln to $y'' - 6y' + 9y = 8e^{3t} + 27t$ is $y = c_1e^{3t} + c_2te^{3t} + 4t^2e^{3t} + 3t + 2$.

or equivalently, $y = e^{3t}(4t^2 + c_2t + c_1) + 3t + 2$.

Step 3: Use initial values to solve for c_1 and c_2:

$y = e^{3t}(4t^2 + c_2t + c_1) + 3t + 2$ implies $y' = 3e^{3t}(4t^2 + c_2t + c_1) + e^{3t}(8t + c_2) + 3$

$y(0) = 5$: $5 = e^0(4(0)^2 + c_2(0) + c_1) + 3(0) + 2$ implies $5 = c_1 + 2$ implies $c_1 = 3$

$y'(0) = 2$: $2 = 3e^0(4(0)^2 + c_2(0) + c_1) + e^0(8(0) + c_2) + 3$

$2 = 3c_1 + c_2 + 3$ implies $c_2 = 2 - 3 - 3c_1 = 2 - 3 - 9 = -10$

Thus solution to IVP is $y = e^{3t}(4t^2 - 10t + 3) + 3t + 2$.

Answer: $y = e^{3t}(4t^2 - 10t + 3) + 3t + 2$