Math 3600 Differential Equations Exam #1
Sept 30, 2016 SHOW ALL WORK

[10] 1a.) Draw the direction field for the following differential equation:

$$y' = (y-2)(y+1)^2$$

[4] 1b.) On the direction field above, draw the solution to the above differential equation that satisfies the initial condition $y(1) = 0$.

[6] 1c.) Does the differential equation whose direction field is given above have any equilibrium solutions? If so, state whether they are stable, semi-stable or unstable.

Equilibrium solution = constant solution, $y = c$ and thus $y' = 0$

$(y-2)(y+1)^2 = 0$ implies $y = 2, -1$

$y = 2$ is unstable, while $y = -1$ is semi-stable.

[15] 2.) Solve the initial value problem **for** $y$: $y' + \frac{3x}{y-4} = 0$, $y(1) = -2$.

$\frac{dy}{dx} = -\frac{3x}{y-4}$

$\int (y-4)dy = \int -3xdx$

$\frac{y^2}{2} - 4y = -\frac{3}{2}x^2$

$y^2 - 8y = -3x^2 + C$

$y^2 - 8y + 3x^2 + C = 0$

$y = \frac{8 \pm \sqrt{64 - 4(3x^2 + C)}}{2} = 4 \pm \sqrt{16 - 3x^2 + C} = y$

$y(1) = -2$: $-2 = 4 \pm \sqrt{16 - 3(1)^2 + C}$ implies $-6 = -\sqrt{16 - 3 + C}$

Note initial value determines sign of $\pm$. In this case, IVP only has a solution when we choose the negative sign. The the IVP in this case means $y = 4 - \sqrt{16 - 3x^2 + C}$ where we determine $C$ below:

$36 = 13 + C$. Thus $C = 36 - 13 = 23$ and $y = 4 - \sqrt{16 - 3x^2 + 23} = 4 - \sqrt{39 - 3x^2}$

Answer: $y = 4 - \sqrt{39 - 3x^2}$

3.) Suppose $y' = y - t + 1, \;\; y(0) = 0.$

Let $\phi_0(t) = 0$ and define $\{\phi_n(t)\}$ by the method of successive approximation (i.e, Picards iteration method). Determine the following:

$y' = f(t, y)$

$\phi_1(t) = \int_0^t f(s, \phi_0(s))ds = \int_0^t f(s, 0)ds = \int_0^t (0 - s + 1)ds =$

$(-\frac{s^2}{2} + s)|_0^t = -\frac{t^2}{2} + t - 0$

[3] 3a) $\phi_1(t) =$ $\underline{\;\; -\frac{t^2}{2} + t \;\;}$

$\phi_2(t) = \int_0^t f(s, \phi_1(s))ds = \int_0^t f(s, -\frac{s^2}{2} + s)ds = \int_0^t (-\frac{s^2}{2} + s - s + 1)ds$

$= \int_0^t (-\frac{s^2}{2} + 1)ds = (-\frac{s^3}{6} + s)|_0^t = -\frac{t^3}{6} + t - 0$

[3] 3b) $\phi_2(t) =$ $\underline{\;\; -\frac{t^3}{6} + t \;\;}$

$\phi_3(t) = \int_0^t f(s, \phi_2(s))ds = \int_0^t f(s, -\frac{s^3}{6} + s)ds = \int_0^t (-\frac{s^3}{6} + s - s + 1)ds$

$= \int_0^t (-\frac{s^3}{6} + 1)ds = (-\frac{s^4}{24} + s)|_0^t = -\frac{t^3}{24} + t - 0$

[3] 3c) $\phi_3(t) =$ $\underline{\;\; -\frac{t^4}{24} + t \;\;}$

[4] 3d) $\phi_n(t) =$ $\underline{\;\; -\frac{t^{n+1}}{(n+1)!} + t \;\;}$

[3] 3e) $lim_{n \to \infty} \phi_n(t) =$ $\underline{\;t\;}$

[2] 3f) Is $\phi(t) = lim_{n \to \infty} \phi_n(t)$ a solution to $y' = y - t + 1, \;\; y(0) = 0$? $\underline{\;\; yes \;\;}$

[2] 3g) Is $\phi(t) = lim_{n \to \infty} \phi_n(t)$ the unique solution to $y' = y - t + 1, \;\; y(0) = 0$? $\underline{\;\; yes \;\;}$

[15] 4a.) Solve $y'' - 8y' + 16y = 0$

Educated guess: $y = e^{rt}$. Then $y' = re^{rt}$ and $y'' = r^2e^{rt}$

Plugging in the guess into our equation:

$r^2e^{rt} - 8re^{rt} + 16e^{rt} = 0$

Since $e^{rt} > 0$, we can divide both sides of the above equation by $re^{rt}$ without loosing any solutions:

$r^2 - 8r + 16 = 0$ implies $(r-4)^2 = 0$ and thus $r = 4$.

Thus $y = e^{4t}$ is a solution. We can check by plugging in (as we did in class for a different example) that $y = te^{4t}$ is also a solution.

Sidenote: $\{e^{4t}, te^{4t}\}$ is a linear independent set and thus a basis for our solution. We can check linear independence by calculating the Wronskian.

Answer: $\underline{\quad y = c_1e^{4t} + c_2te^{4t} \quad}$

[15] 4b.) Solve $y'' - y' + 3y = 0$

Educated guess: $y = e^{rt}$. Then $y' = re^{rt}$ and $y'' = r^2e^{rt}$

Plugging in the guess into our equation:

$r^2e^{rt} - re^{rt} + 3e^{rt} = 0$

Since $e^{rt} > 0$, we can divide both sides of the above equation by $re^{rt}$ without loosing any solutions:

$r^2 - r + 3 = 0$ implies $r = \frac{1 \pm \sqrt{1-4(3)}}{2} = \frac{1 \pm \sqrt{11}}{2} = \frac{1 \pm i\sqrt{11}}{2}$.

Answer: $\underline{\quad y = c_1e^{\frac{t}{2}}cos(\frac{\sqrt{11}}{2}t) + c_2e^{\frac{t}{2}}sin(\frac{\sqrt{11}}{2}t) \quad}$

[15] 5.) Let $y = y_1(t)$ be a solution of $y' + p(t)y = 0$ and let $y = y_2(t)$ be a solution of $y' + p(t)y = g(t)$. Show that $y = y_1(t) + y_2(t)$ is a solution of $y' + p(t)y = g(t)$.

Proof: Since $y = y_1(t)$ is a solution of $y' + p(t)y = 0$, we know that $y_1' + p(t)y_1 = 0$.

Since $y = y_2(t)$ is a solution of $y' + p(t)y = g(t)$, $\quad y_2' + p(t)y_2 = g(t)$

Claim: $y = y_1(t) + y_2(t)$ is a solution of $y' + p(t)y = g(t)$.

We will plug $y = y_1(t) + y_2(t)$ into the LHS to determine that the LHS = RHS:

$$(y_1(t) + y_2(t))' + p(t)(y_1(t) + y_2(t)) = y_1'(t) + y_2'(t) + p(t)y_1(t) + p(t)y_2(t)$$

$$= [y_1'(t) + p(t)y_1(t)] + [y_2'(t) + p(t)y_2(t)] = 0 + g(t) = g(t)$$

Hence $y = y_1(t) + y_2(t)$ is a solution of $y' + p(t)y = g(t)$.

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Alternate proof: Since $y = y_1(t)$ is a solution of $y' + p(t)y = 0$, we know that

$$y_1' + p(t)y_1 = 0 \quad (1).$$

Since $y = y_2(t)$ is a solution of $y' + p(t)y = g(t)$.

$$y_2' + p(t)y_2 = g(t) \quad (2).$$

If we add equations (1) and (2), we obtain:

$$[y_1'(t) + p(t)y_1(t)] + [y_2'(t) + p(t)y_2(t)] = 0 + g(t)$$

Thus $y_1'(t) + y_2'(t) + p(t)y_1(t) + p(t)y_2(t) = g(t)$

and $(y_1(t) + y_2(t))' + p(t)(y_1(t) + y_2(t)) = g(t)$

Hence $y = y_1(t) + y_2(t)$ is a solution of $y' + p(t)y = g(t)$.

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Alternate proof:
Claim: $L(f) = f' + pf$ is a linear function where $f$ and $p$ are functions of $t$.

Proof of claim: Let $a, b$ be constants and $f, g$ be functions of $t$.

$L(af + bg) = (af + bg)' + p(af + bg) = af' + bg' + paf + pbg = af' + paf + bg' + pbg = [a(f' + pf)] + [b(g' + pg)] = L(f) + L(g)$

We will now show that $y = y_1(t) + y_2(t)$ is a solution of $y' + p(t)y = g(t)$:

Since $y = y_1(t)$ is a solution of $y' + p(t)y = 0, \quad L(y_1) = 0$.

Since $y = y_2(t)$ is a solution of $y' + p(t)y = g(t), \;\; L(y_2) = g(t)$

$L(y_1 + y_2) = L(y_1) + L(y_2) = 0 + g(t) = g(t)$. Thus $y = y_1(t) + y_2(t)$ is a solution of $y' + p(t)y = g(t)$.

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Note similar proofs would show that $y = cy_1(t) + y_2(t)$ is a solution of $y' + p(t)y = g(t)$ for any constant $c$.