

Suppose an object moves in the 2D plane (the x_1, x_2 plane) so that it is at the point $(x_1(t), x_2(t))$ at time t . Suppose the object's velocity is given by

$$\begin{aligned} \dot{x}_1(t) &= ax_1 + bx_2 \\ \dot{x}_2(t) &= cx_1 + dx_2 \end{aligned}$$

Or in matrix form $\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

To solve, find eigenvalues and corresponding eigenvectors:

$$\begin{vmatrix} a-r & b \\ c & d-r \end{vmatrix} = (a-r)(d-r) - bc = r^2 - (a+d)r + ad - bc = 0.$$

Thus $r = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$

Case 1: $(a+d)^2 - 4(ad-bc) > 0$ Two real e. values
 Hence the general solutions is $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} v_1 \\ w_1 \end{pmatrix} e^{r_1 t} + c_2 \begin{pmatrix} v_2 \\ w_2 \end{pmatrix} e^{r_2 t}$



Case 1c: $r_2 < 0 < r_1$



Case 2: $(a+d)^2 - 4(ad-bc) = 0$ | repeated

Case 2i: Two independent eigenvectors:

The general solution is $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} v_1 \\ w_1 \end{pmatrix} e^{rt} + c_2 \begin{pmatrix} v_2 \\ w_2 \end{pmatrix} e^{rt}$

Case 2ii: One independent eigenvectors:

The general solution is $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} v_1 \\ w_1 \end{pmatrix} e^{rt} + c_2 \left[\begin{pmatrix} v_1 \\ w_1 \end{pmatrix} t + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \right] e^{rt}$

Case 2a: $r > 0$

Case 2b: $r < 0$

Case 3: $(a+d)^2 - 4(ad-bc) < 0$. I.e., $r = \lambda \pm i\mu$

Suppose the eigenvector corresponding to this eigenvalue is

$$\begin{pmatrix} v_1 + iw_1 \\ v_2 + iw_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + i \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

Then general solution is $\vec{x} = c_1 (\vec{v} + i\vec{w}) e^{\lambda t}$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} v_1 \cos(\mu t) - w_1 \sin(\mu t) \\ v_2 \cos(\mu t) - w_2 \sin(\mu t) \end{pmatrix} e^{\lambda t} + c_2 \begin{pmatrix} v_1 \sin(\mu t) + w_1 \cos(\mu t) \\ v_2 \sin(\mu t) + w_2 \cos(\mu t) \end{pmatrix} e^{\lambda t}$$

Case 3a: $\lambda > 0$

Case 3a: $\lambda < 0$

Case 3a: $\lambda = 0$

