With this we can now solve an IVP that involves a Dirac Delta function.

Example 1 Solve the following IVP.

$$y'' + 2y' - 15y = 6\delta(t-9), \qquad y(0) = -5 \quad y'(0) = 7$$

Solution

As with all previous problems we'll first take the Laplace transform of everything in the differential equation and apply the initial conditions.

$$s^2Y(s) - sy(0) - y'(0) + 2\big(sY(s) - y(0)\big) - 15Y(s) = 6\mathbf{e}^{-9s}$$

$$\left(s^2 + 2s - 15\right)Y(s) + 5s + 3 = 6\mathbf{e}^{-9s}$$

Now solve for *Y(s)*.

$$\begin{aligned} Y(s) &= \frac{6\mathbf{e}^{-9s}}{(s+5)(s-3)} - \frac{5s+3}{(s+5)(s-3)} \\ &= 6\mathbf{e}^{-9s}F(s) - G(s) \end{aligned}$$

We'll leave it to you to verify the partial fractions and their inverse transforms are,

$$F(s) = \frac{1}{(s+5)(s-3)} = \frac{\frac{1}{8}}{s-3} - \frac{\frac{1}{8}}{s+5}$$

$$f(t) = \frac{1}{8}\mathbf{e}^{3t} - \frac{1}{8}\mathbf{e}^{-5t}$$

$$G(s) = \frac{5s+3}{(s+5)(s-3)} = \frac{\frac{9}{4}}{s-3} + \frac{\frac{11}{4}}{s+5}$$

$$g(t) = \frac{9}{4}\mathbf{e}^{3t} + \frac{11}{4}\mathbf{e}^{-5t}$$

The solution is then,

$$Y(s) = 6\mathbf{e}^{-9s}F(s) - G(s)$$

$$y(t) = 6u_9(t)\, f(t-9) - g(t)$$

where, *f(t)* and *g(t)* are defined above.

Example 2 Solve the following IVP.

$$2y'' + 10y = 3u_{12}(t) - 5\delta(t-4), \qquad y(0) = -1 \quad y'(0) = -2$$

Solution

Take the Laplace transform of everything in the differential equation and apply the initial conditions.

$$2\big(s^2Y(s) - sy(0) - y'(0)\big) + 10Y(s) = \frac{3\mathbf{e}^{-12s}}{s} - 5\mathbf{e}^{-4s}$$

$$\left(2s^2 + 10\right)Y(s) + 2s + 4 = \frac{3\mathbf{e}^{-12s}}{s} - 5\mathbf{e}^{-4s}$$

Now solve for *Y(s)*.

$$Y(s) = \frac{3\mathbf{e}^{-12s}}{s(2s^2+10)} - \frac{5\mathbf{e}^{-4s}}{2s^2+10} - \frac{2s+4}{2s^2+10}$$
$$= 3\mathbf{e}^{-12s}F(s) - 5\mathbf{e}^{-4s}G(s) - H(s)$$

We'll need to partial fraction the first function. The remaining two will just need a little work and they'll be ready. I'll leave the details to you to check.

$$F(s) = \frac{1}{s(2s^2+10)} = \frac{1}{10}\frac{1}{s} - \frac{1}{10}\frac{s}{s^2+5}$$
$$f(t) = \frac{1}{10} - \frac{1}{10}\cos\left(\sqrt{5}\,t\right)$$
$$g(t) = \frac{1}{2\sqrt{5}}\sin\left(\sqrt{5}\,t\right)$$
$$h(t) = \cos\left(\sqrt{5}\,t\right) + \frac{2}{\sqrt{5}}\sin\left(\sqrt{5}\,t\right)$$

The solution is then,

$$Y(s) = 3\mathbf{e}^{-12s}F(s) - 5\mathbf{e}^{-4s}G(s) - H(s)$$
$$y(t) = 3u_{12}(t)\,f(t-12) - 5u_4(t)\,g(t-4) - h(t)$$

where, *f(t)*,*g(t)* and *h(t)* are defined above.

So, with the exception of the new function these work the same way that all the problems that we've seen to this point work. Note as well that the exponential was introduced into the transform by the Dirac Delta function, but once in the transform it doesn't matter where it came from. In other words, when we went to the inverse transforms it came back out as a Heaviside function.

Before proceeding to the next section let's take a quick side trip and note that we can relate the Heaviside function and the Dirac Delta function. Start with the following integral.

$$\int_{-\infty}^{t} \delta(u-a)\,du = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t > a \end{cases}$$

However, this is precisely the definition of the Heaviside function. So,

$$\int_{-\infty}^{t} \delta(u-a)\,du = u_a(t)$$

Now, recalling the Fundamental Theorem of Calculus, we get,

$$u_a'(t) = \frac{d}{dt}\left(\int_{-\infty}^{t} \delta(u-a)\,du\right) = \delta(t-a)$$

So, the derivative of the Heaviside function is the Dirac Delta function.