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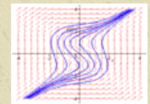
Ch 2.2: Separable Equations

- In this section we examine a subclass of linear and nonlinear first order equations. Consider the first order equation
$$\frac{dy}{dx} = f(x, y)$$
- We can rewrite this in the form
$$M(x, y) + N(x, y)\frac{dy}{dx} = 0$$
- For example, let $M(x,y) = - f(x,y)$ and $N(x,y) = 1$. There may be other ways as well. In differential form,
$$M(x, y)dx + N(x, y)dy = 0$$
- If M is a function of x only and N is a function of y only, then
$$M(x)dx + N(y)dy = 0$$
- In this case, the equation is called **separable**.

Example 1: Solving a Separable Equation

- Solve the following first order nonlinear equation:
$$\frac{dy}{dx} = \frac{x^2+1}{y^2-1}$$
- Separating variables, and using calculus, we obtain
$$\left(y^2-1\right)dy = \left(x^2+1\right)dx$$
$$\int\left(y^2-1\right)dy = \int\left(x^2+1\right)dx$$
$$\frac{1}{3}y^3 - y = \frac{1}{3}x^3 + x + C$$
$$y^3 - 3y = x^3 + 3x + C$$



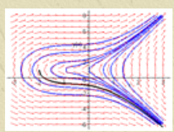
- The equation above defines the solution y implicitly. A graph showing the direction field and implicit plots of several integral curves for the differential equation is given above.

Example 2: Implicit and Explicit Solutions (1 of 4)

- Solve the following first order nonlinear equation:
$$\frac{dy}{dx} = \frac{3x^2+4x+2}{2(y-1)}$$
- Separating variables and using calculus, we obtain
$$2(y-1)dy = \left(3x^2+4x+2\right)dx$$
$$2\int(y-1)dy = \int\left(3x^2+4x+2\right)dx$$
$$y^2 - 2y = x^3 + 2x^2 + 2x + C$$
- The equation above defines the solution y implicitly. An explicit expression for the solution can be found in this case:
$$y^2 - 2y - \left(x^3+2x^2+2x+C\right) = 0 \Rightarrow y = \frac{2 \pm \sqrt{4+4\left(x^3+2x^2+2x+C\right)}}{2}$$
$$y = 1 \pm \sqrt{x^3+2x^2+2x+C}$$

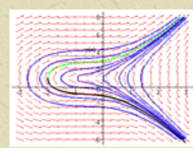
Example 2: Initial Value Problem (2 of 4)

- Suppose we seek a solution satisfying $y(0) = -1$. Using the implicit expression of y, we obtain
$$y^2 - 2y = x^3 + 2x^2 + 2x + C$$
$$(-1)^2 - 2(-1) = C \Rightarrow C = 3$$
- Thus the implicit equation defining y is
$$y^2 - 2y = x^3 + 2x^2 + 2x + 3$$
- Using explicit expression of y,
$$y = 1 \pm \sqrt{x^3+2x^2+2x+C}$$
$$-1 = 1 \pm \sqrt{C} \Rightarrow C = 4$$
- It follows that
$$y = 1 - \sqrt{x^3+2x^2+2x+4}$$



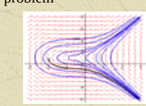
Example 2: Initial Condition $y(0) = 3$ (3 of 4)

- Note that if initial condition is $y(0) = 3$, then we choose the positive sign, instead of negative sign, on square root term:
$$y = 1 + \sqrt{x^3+2x^2+2x+4}$$



Example 2: Domain (4 of 4)

- Thus the solutions to the initial value problem
$$\frac{dy}{dx} = \frac{3x^2+4x+2}{2(y-1)}, \quad y(0) = -1$$
are given by
$$y^2 - 2y = x^3 + 2x^2 + 2x + 3 \quad \text{(implicit)}$$
$$y = 1 - \sqrt{x^3+2x^2+2x+4} \quad \text{(explicit)}$$



- From explicit representation of y, it follows that
$$y = 1 - \sqrt{x^2(x+2) + 2(x+2)} = 1 - \sqrt{(x+2)\left(x^2+2\right)}$$
and hence domain of y is $(-2, \infty)$. Note $x = -2$ yields $y = 1$, which makes denominator of dy/dx zero (vertical tangent).
- Conversely, domain of y can be estimated by locating vertical tangents on graph (useful for implicitly defined solutions).

Example 3: Implicit Solution of Initial Value Problem (1 of 2)

- Consider the following initial value problem:
$$y' = \frac{y\cos x}{1+3y^3}, \quad y(0) = 1$$
- Separating variables and using calculus, we obtain
$$\frac{1+3y^3}{y}dy = \cos x\,dx$$
$$\int\left(\frac{1}{y} + 3y^2\right)dy = \int\cos x\,dx$$
$$\ln|y| + y^3 = \sin x + C$$
- Using the initial condition, it follows that
$$\ln y + y^3 = \sin x + 1$$

Example 3: Graph of Solutions (2 of 2)

- Thus
$$y' = \frac{y\cos x}{1+3y^3}, \quad y(0) = 1 \Rightarrow \ln y + y^3 = \sin x + 1$$
- The graph of this solution (black), along with the graphs of the direction field and several integral curves (blue) for this differential equation, is given below.