9.3 Locally linear systems.

Just like in Calc 1, we are interested in critical points. In this class, these are equilibrium solutions, $(x(t), y(t)) = (x_0, y_0)$, i.e constant solutions in (x, y, t) space which thus project to a point in the (x, y) phase plane.

Just like in Calc 1, we will use linear approximations to determine what solutions look like near the critical point in the (x, y) phase plane.

For Calc 1, linear approximation of f at x_0 = tangent line. Thus by Taylor's theorem:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \eta(x) \text{ where } \eta \text{ is error term.}$$

For a function of 2 variables, if we fix $y = y_0$, then

$$F(x, y_0) = F(x_0, y_0) + F_x(x_0, y_0)(x - x_0) + \eta(x, y_0)$$

The visualization when x and y are functions of t when given an autonomous system of differential equations $\frac{dx}{dt} = F(x, y)$, $\frac{dy}{dt} = G(x, y)$ is much more interesting than the calc 1 case. It is also different than the case when z is a function of t and s.

Goal: Find linear approximation for autonomous system of 1^{st} order D. E.:

$$x' = F(x, y), \; y' = G(x, y)$$

And use this linear approximation to determine what a trajectory $(x(t), y(t))$, $t \in (a, b)$ looks like near a critical point.

We will approximate the two functions, $z_1 = F(x, y)$ and $z_2 = G(x, y)$. By Taylor's theorem, if F and G have continuous partial derivatives up to order two, then

$$F(x, y) = F(x_0, y_0) + F_x(x_0, y_0)(x - x_0) + F_y(x_0, y_0)(y - y_0) + \eta_1(x, y)$$

$$G(x, y) = G(x_0, y_0) + G_x(x_0, y_0)(x - x_0) + G_y(x_0, y_0)(y - y_0) + \eta_2(x, y)$$

$$\text{where } lim_{(x,y)\to(x_0,y_0)} \frac{\eta_i(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}} = 0$$

Since (x_0, y_0) is a critical point, $F(x_0, y_0) = 0$ and $G(x_0, y_0) = 0$. Thus

$$\frac{d}{dt}\begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix} = \frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} F(x, y) \\ G(x, y) \end{bmatrix} = \begin{bmatrix} F_x(x_0, y_0) & F_y(x_0, y_0) \\ G_x(x_0, y_0) & G_y(x_0, y_0) \end{bmatrix} \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix} + \begin{bmatrix} \eta_1(x, y) \\ \eta_2(x, y) \end{bmatrix}$$

Note $\begin{bmatrix} F_x(x_0, y_0) & F_y(x_0, y_0) \\ G_x(x_0, y_0) & G_y(x_0, y_0) \end{bmatrix}$ is a matrix with constant entries.

Let $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}$

Then a **translated** linear approximation of the above matrix D. E. equation is

$$\frac{d}{dt}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} F_x(x_0, y_0) & F_y(x_0, y_0) \\ G_x(x_0, y_0) & G_y(x_0, y_0) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Note by substituting $u_1 = x - x_0$, $u_2 = y - y_0$, we have **translated** our critical point (x_0, y_0) in the x, y-phase plane to the origin (0, 0) in the u_1, u_2-phase plane.

Defn: $J(x, y) = \begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix}$ is the *Jacobian matrix* of F and G.

Defn: $det(J(x, y))$ is the *Jacobian.*

We will restrict to the case $det(J(x_0, y_0)) \neq 0$ for the following reasons:

Just like in Calc 1, we will restrict to only looking at isolated critical points. That is we can find an $\epsilon-$ball around (x_0, y_0) such that this $\epsilon-$ball does not contain any other critical points. That is,
(x_0, y_0) is the unique critical point in the region defined by $\sqrt{(x - x_0)^2 + (y - y_0)^2} < \epsilon$

In other words, if $\mathbf{x}' = A\mathbf{x}$ is a linear approximation to the autonomous system of first order D. E.: $x' = F(x, y)$, $y' = G(x, y)$ near (x_0, y_0), then we will restrict to the case when $det(A) \neq 0$.

The stability of the critical point $(x, y) = (x_0, y_0)$ of the system $x' = F(x, y)$, $y' = G(x, y)$ will depend on both

(1.) The stability of the critical point $(x, y) = (0, 0)$ of its translated linear approximation $\mathbf{x}' = A\mathbf{x}$.

(2.) How stable is the stability of the critical point $(x, y) = (0, 0)$ of its translated linear approximation $\mathbf{x}' = A\mathbf{x}$.

Example: $x' = x - xy, \;\; y' = -y + xy$ Eqn (*)

critcal points: $x' = x - xy = x(1-y) = 0$ implies $x = 0$ or $y = 1$,
$y' = -y + xy = y(-1+x) = 0$ implies $y = 0$ or $x = 1$. Break into cases using 1rst equation (or any equation of your choice)

Case 1: If $x = 0$, then $y = 0$ by second case. Thus (0, 0) is a critical point.
Case 2: If $y = 1$, then $x = 1$ by second case. Thus (1, 1) is a critical point.

Jacobian matrix: $\begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix} = \begin{bmatrix} 1-y & -x \\ y & -1+x \end{bmatrix}$

Case 1: $(x_0, y_0) = (0, 0)$.

The linear approximation to non-linear differential equation (*) is

$$\mathbf{x}' = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{x}$$

Determine stability of critical point (0, 0) of linear approximation:

Short method: $eigenvalues = 1, -1$, thus (0, 0) is an unstable saddle point of the linear approximation. If we slightly perturb $(p, q) = (a+d, ad-bc) = (0, -1)$, we still have an unstable saddle point.

Thus $(x_0, y_0) = (0, 0)$ is an unstable saddle point of the nonlinear system of D.E., $x' = x - xy, \;\; y' = -y + xy$

Longer method: $det(A - rI) = \begin{vmatrix} 1-r & 0 \\ 0 & -1-r \end{vmatrix} = (1-r)(-1-r) = r^2 - 1 = 0$

Thus $r = \frac{0 \pm \sqrt{0^2 - 4(-1)}}{2} = \frac{p \pm \sqrt{p^2 - 4(q)}}{2}$ where $r^2 - pr + q = 0$. I.e., $p = 0$, $q = -1$. See figure 9.1.9 in your text.

Thus (0, 0) is an unstable saddle point of the linear approximation. If we slightly perturb the linear approximation differential equation, then we will still have one positive and one negative eigenvalue (since values close to the eigenvalue 1 will still be positive and values close to the eigenvalue -1 will still be negative. Thus we still have an unstable saddle point. Similarly, in figure 9.1.9 when $p = 0$, $q = -1$, we get an unstable saddle point for small perturbations of p and q.

Thus $(x_0, y_0) = (0, 0)$ is an unstable saddle point of nonlinear system of D.E.,

$$x' = x - xy, \;\; y' = -y + xy$$

Case 2: $(x_0, y_0) = (1, 1)$.

The linear translated approximation to non-linear differential equation (*) is

$$\mathbf{x}' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{x}$$

$$det(A - rI) = \begin{vmatrix} -r & -1 \\ 1 & -r \end{vmatrix} = r^2 + 1 = 0. \text{ Thus } r = \frac{0 \pm \sqrt{0^2 - 4(1)}}{2} = \pm i.$$

Thus (0, 0) is a stable center point of the linear translated approximation.

If we take complex numbers, $a \pm bi$ close to the eigenvalues $r = \pm i$, then b will be close to 1 and thus positive, but a will be close to 0 and thus could be positive or negative or 0.

Alternatively, see figure 9.1.9 in your text where $p = a + d = 0$ and $q = ad - bc = 1$.

Hence for the nonlinear equation (*), the critical point (1, 1) is one of the following
1.) stable center point
2.) unstable spiral point
3.) asymptotically stable spiral point.