7.7: The matrix exponential:

Defn: $\Sigma_{n=0}^{\infty} b_k x^k = \lim_{n\to\infty} \Sigma_{n=0}^{k} b_k x^k = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + ...$

Taylor's Theorem: If f is analytic at 0, then for small x (i.e., x near 0),

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = f(0) + f'(0)x + \tfrac{f''(0)}{2}x^2 + \tfrac{f'''(0)}{6}x^3 + ...$$

Example:

$e^t = \Sigma_{n=0}^{\infty} \frac{t^k}{k!}$ and thus $e^{at} = \Sigma_{n=0}^{\infty} \frac{a^k t^k}{k!} = 1 + \Sigma_{n=1}^{\infty} \frac{a^k t^k}{k!}$ for t near 0.

Definition: Let A be an $n \times n$ matrix. Then the **matrix exponential** of A is

$$exp(At) = e^{At} = I + \Sigma_{n=1}^{\infty} \frac{A^k t^k}{k!}$$

where I is the $n \times n$ identity matrix.

Note $e^{A(0)} = I + \Sigma_{n=1}^{\infty} \frac{A^k 0^k}{k!} = I$

Note $[e^{At}]' = \Sigma_{n=1}^{\infty} \frac{kA^k t^{k-1}}{k!} = \Sigma_{n=1}^{\infty} \frac{A^k t^{k-1}}{(k-1)!} = \Sigma_{n=0}^{\infty} \frac{A^{k+1} t^k}{k!}$

$$= A + \Sigma_{n=1}^{\infty} \frac{A^{k+1} t^k}{k!} = A(I + \Sigma_{n=1}^{\infty} \frac{A^k t^k}{k!}) = Ae^{At}$$

Thus $[e^{At}]' = Ae^{At}$ and $e^{A(0)} = I$

Thus e^{At} is the solution to the IVP

$$M' = AM,\ M(0) = I$$

where M is an $n \times n$ matrix.

Let Ψ be a fundamental matrix for $\mathbf{x}' = A\mathbf{x}$

Example: Suppose Ψ is the 2×2 matrix $[\mathbf{f_1} \quad \mathbf{f_2}]$.

Thus $\mathbf{f_1}$ and $\mathbf{f_2}$ are solutions to $\mathbf{x}' = A\mathbf{x}$.

Hence $\mathbf{f_1}' = A\mathbf{f_1}$ and $\mathbf{f_2}' = A\mathbf{f_2}$

Thus $A[\mathbf{f_1} \quad \mathbf{f_2}] =$

Since $[\mathbf{f_1} \quad \mathbf{f_2}]' = A[\mathbf{f_1} \quad \mathbf{f_2}]$,

$$\Psi(t) = [\mathbf{f_1(t)} \quad \mathbf{f_2(t)}] \text{ is the general solution to } M' = AM$$

Let $\Phi(t) = \Psi(t)[\Psi(0)]^{-1}$. Then $\Phi(0) = \Psi(0)[\Psi(0)]^{-1} = I$.

Note $\Phi(t) = \Psi(t)[\Psi(0)]^{-1}$ is also a solution to $M' = AM$:

$$(\Psi(t)[\Psi(0)]^{-1})' = \Psi'(t)[\Psi(0)]^{-1} = A\Psi(t)[\Psi(0)]^{-1}$$

Thus Φ is the solution to the IVP

$$M' = AM, \ M(0) = I$$

Since solution to IVP is unique (assuming entries of A are continuous functions), $\Phi = e^{At}$

Theorem: $e^{At} = \Phi(t) = \Psi(t)[\Psi(0)]^{-1}$ where Ψ is a fundamental matrix for $\mathbf{x}' = A\mathbf{x}$

Example: Calculate $exp(\begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} t) = e^{\begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} t}$

Solve $\mathbf{x}'(t) = \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} \mathbf{x}(\mathbf{t})$

From previous work, the general solution is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{7t} + c_2 \begin{bmatrix} 3 \\ -2 \end{bmatrix} e^{-t}$$

Thus a fundamental matrix is $\Psi(t) = \begin{bmatrix} e^{7t} & 3e^{-t} \\ 2e^{7t} & -2e^{-t} \end{bmatrix}$

A "better" fundamental matrix is $\Phi(t) = \Psi(t)[\Psi(0)]^{-1}$

$\Psi(0) = \begin{bmatrix} 1 & 3 \\ 2 & -2 \end{bmatrix}$. Thus $[\Psi(0)]^{-1} = \begin{bmatrix} 1 & 3 \\ 2 & -2 \end{bmatrix}^{-1} = (-\frac{1}{8}) \begin{bmatrix} -2 & -3 \\ -2 & 1 \end{bmatrix}$

$$\Phi(t) = \Psi(t)[\Psi(0)]^{-1} = (-\tfrac{1}{8}) \begin{bmatrix} e^{7t} & 3e^{-t} \\ 2e^{7t} & -2e^{-t} \end{bmatrix} \begin{bmatrix} -2 & -3 \\ -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{4}e^{7t} + \frac{3}{4}e^{-t} & \frac{3}{8}e^{7t} - \frac{3}{8}e^{-t} \\ \frac{1}{2}e^{7t} - \frac{1}{2}e^{-t} & \frac{3}{4}e^{7t} + \frac{1}{4}e^{-t} \end{bmatrix}$$

Thus $exp(\begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} t) = e^{\begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} t} = \begin{bmatrix} \frac{1}{4}e^{7t} + \frac{3}{4}e^{-t} & \frac{3}{8}e^{7t} - \frac{3}{8}e^{-t} \\ \frac{1}{2}e^{7t} - \frac{1}{2}e^{-t} & \frac{3}{4}e^{7t} + \frac{1}{4}e^{-t} \end{bmatrix}$

Solve IVP: $\mathbf{x}' = A\mathbf{x}$, $\mathbf{x}(0) = \mathbf{e}$

Observe if $\Phi(t) = [\mathbf{f_1} \;\; \mathbf{f_2}]$, then general solution is

$$\mathbf{x(t)} = c_1\mathbf{f_1}(t) + c_2\mathbf{f_2}(t) = [\mathbf{f_1}(t) \;\; \mathbf{f_2}(t)] \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \Phi(t) \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\mathbf{x(0)} = c_1\mathbf{f_1}(0) + c_2\mathbf{f_2}(0) = [\mathbf{f_1}(0) \;\; \mathbf{f_2}(0)] \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$= \Phi(0) \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Theorem: If $e^{At} = \Phi(t) = \Psi(t)[\Psi(0)]^{-1}$ where Ψ is a fundamental matrix for $\mathbf{x}' = A\mathbf{x}$

or equivalently, if Φ is the solution to IVP $M' = AM$, $M(0) = I$,

Then solution to IVP $\mathbf{x}' = A\mathbf{x}$, $\mathbf{x}(0) = \mathbf{e}$ is $\mathbf{x} = \Phi\mathbf{e}$

Example: Solve IVP: $\mathbf{x}'(t) = \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} \mathbf{x(t)}$, $\quad \mathbf{x}(0) = \begin{bmatrix} 17 \\ 92 \end{bmatrix}$

Answer: $\mathbf{x} = \begin{bmatrix} \frac{1}{4}e^{7t} + \frac{3}{4}e^{-t} & \frac{3}{8}e^{7t} - \frac{3}{8}e^{-t} \\ \frac{1}{2}e^{7t} - \frac{1}{2}e^{-t} & \frac{3}{4}e^{7t} + \frac{1}{4}e^{-t} \end{bmatrix} \begin{bmatrix} 17 \\ 92 \end{bmatrix}$

That is, $\mathbf{x} = 17 \begin{bmatrix} \frac{1}{4}e^{7t} + \frac{3}{4}e^{-t} \\ \frac{1}{2}e^{7t} - \frac{1}{2}e^{-t} \end{bmatrix} + 92 \begin{bmatrix} \frac{3}{8}e^{7t} - \frac{3}{8}e^{-t} \\ \frac{3}{4}e^{7t} + \frac{1}{4}e^{-t} \end{bmatrix}$

Note: This only works if you use the fundamental matrix $\Phi(t)$ where $\Phi(0) = I$.

7.5/7.8: Suppose the solution to $\mathbf{x}' = A\mathbf{x}$ is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} e^{r_1 t} + c_2 \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} e^{r_2 t}$$

where $\mathbf{v}, \mathbf{w}$ are linearly independent.

Then $$\frac{x_2(t)}{x_1(t)} = \frac{c_1 v_2 e^{r_1 t} + c_2 w_2 e^{r_2 t}}{c_1 v_1 e^{r_1 t} + c_2 w_1 e^{r_2 t}} = \frac{c_1 v_2 + c_2 w_2 e^{(r_2 - r_1)t}}{c_1 v_1 + c_2 w_1 e^{(r_2 - r_1)t}}$$

If $c_2 = 0$, then $\frac{x_2(t)}{x_1(t)} = \frac{v_2}{v_1}$

If $c_1 = 0$, then $\frac{x_2(t)}{x_1(t)} = \frac{w_2}{w_1}$

Suppose $r_1 > r_2$ and $c_1 c_2 \neq 0$,

Then $$\lim_{t \to \infty} \frac{x_2(t)}{x_1(t)} = \lim_{t \to \infty} \frac{c_1 v_2 + c_2 w_2 e^{(r_2 - r_1)t}}{c_1 v_1 + c_2 w_1 e^{(r_2 - r_1)t}} = \frac{v_2}{v_1}$$

Similarly $$\lim_{t \to -\infty} \frac{x_2(t)}{x_1(t)} = \lim_{t \to -\infty} \frac{c_1 v_2 e^{(r_1 - r_2)t} + c_2 w_2}{c_1 v_1 e^{(r_1 - r_2)t} + c_2 w_1} = \frac{w_2}{w_1}$$

$r_1 > 0 > r_2$ $\qquad$ $r_1 > r_2 > 0$ $\qquad$ $0 > r_1 > r_2$

Repeated root case with 2 linearly independent eigenvectors:
If $r_1 = r_2$, then

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = e^{r_1 t}\left(c_1 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + c_2 \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}\right) = e^{r_1 t}\begin{bmatrix} c_1 v_1 + c_2 w_1 \\ c_1 v_2 + c_2 w_2 \end{bmatrix}$$

Then $\dfrac{x_2(t)}{x_1(t)} = \dfrac{e^{r_1 t}(c_1 v_2 + c_2 w_2)}{e^{r_1 t}(c_1 v_1 + c_2 w_1)} = \dfrac{c_1 v_2 + c_2 w_2}{c_1 v_1 + c_2 w_1}$

Repeated root case, $r_1 = r_2$, with only 1 linearly independent eigenvector, $\mathbf{v}$

One solution: $\mathbf{x} = \mathbf{v}e^{r_1 t}$

Need 2nd solution, Guess $\mathbf{x} = t\mathbf{v}e^{r_1 t} + \mathbf{w}e^{r_1 t}$

Then $\mathbf{x}' = r_1 t\mathbf{v}e^{r_1 t} + \mathbf{v}e^{r_1 t} + r_1\mathbf{w}e^{r_1 t}$

Plug into $\mathbf{x}' = A\mathbf{x}$: $\quad r_1 t\mathbf{v}e^{r_1 t} + \mathbf{v}e^{r_1 t} + r_1\mathbf{w}e^{r_1 t} = A(t\mathbf{v}e^{r_1 t} + \mathbf{w}e^{r_1 t})$

$$r_1 t\mathbf{v}e^{r_1 t} + \mathbf{v}e^{r_1 t} + r_1\mathbf{w}e^{r_1 t} = te^{r_1 t}A\mathbf{v} + e^{r_1 t}A\mathbf{w}$$

$$r_1 t\mathbf{v}e^{r_1 t} + \mathbf{v}e^{r_1 t} + r_1\mathbf{w}e^{r_1 t} = te^{r_1 t}r_1\mathbf{v} + e^{r_1 t}A\mathbf{w}$$

$$r_1 t\mathbf{v} + \mathbf{v} + r_1\mathbf{w} = tr_1\mathbf{v} + A\mathbf{w}$$

$$\mathbf{v} + r_1\mathbf{w} = A\mathbf{w}$$

Thus $\mathbf{v} = A\mathbf{w} - r_1\mathbf{w} =$

Hence $\mathbf{v} =$

Definition: If r_1 is an eigenvalue with eigenvector $\mathbf{v}$, then $\mathbf{w}$ is a **generalized eigenvector** corresponding to eigenvalue r_1, if $\mathbf{w} \neq 0$ and

$$(A - r_1 I)\mathbf{w} = \mathbf{v}$$

General solution:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} e^{r_1 t} + c_2 \left(t \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \right) e^{r_1 t}$$

or equivalently,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = e^{r_1 t} \left(c_1 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + c_2 \left(t \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \right) \right)$$

Then

$$\frac{x_2(t)}{x_1(t)} = \frac{e^{r_1 t}(c_1 v_2 + c_2(t v_2 + w_2))}{e^{r_1 t}(c_1 v_1 + c_2(t v_1 + w_1))} = \frac{c_1 v_2 + c_2 w_2 + c_2 t v_2}{c_1 v_1 + c_2 w_1 + c_2 t v_1}$$

If $c_2 = 0$, then $\frac{x_2(t)}{x_1(t)} = \frac{v_2}{v_1}$.

No other constant slopes, but $\displaystyle\lim_{t \to \pm\infty} \frac{x_2(t)}{x_1(t)} = \frac{v_2}{v_1}$

Example: Solve $\mathbf{x}' = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} \mathbf{x}$

Repeated root eigenvalue $r = 3$

To find eigenvector corresponding to eigenvalue 3, solve $(A - 3I)\mathbf{x} = \mathbf{0}$:

$$A - 3I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Thus x_1 is free and $x_2 = 0$. Thue $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector corresponding to eigenvalue 3.

To find generalized eigenvector w corresponding to eigenvalue 3 and eigenvector v,

$$\textbf{solve } (A - 3I)\mathbf{w} = \mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Solve $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Thus can choose $\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Hence general solution is $\mathbf{x} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{3t} + c_2 \left(t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) e^{3t}$