Solve $\mathbf{X}'(t) = \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} \mathbf{X(t)}$

Step 1. Find eigenvalues:

$$A - \lambda I = \begin{vmatrix} 1-\lambda & 3 \\ 4 & 5-\lambda \end{vmatrix} = (1-\lambda)(5-\lambda) - 12$$
$$= \lambda^2 - 6\lambda + 5 - 12 = \lambda^2 - 6\lambda - 7 = (\lambda - 7)(\lambda + 1) = 0$$

Thus $\lambda = 7, -1$

Step 2. Find eigenvectors:

$$\lambda = 7: \quad A - 7I = \begin{bmatrix} 1-7 & 3 \\ 4 & 5-7 \end{bmatrix} = \begin{bmatrix} -6 & 3 \\ 4 & -2 \end{bmatrix}$$

Note $\begin{bmatrix} -6 & 3 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Note the dimension of the nullspace of $\begin{bmatrix} -6 & 3 \\ 4 & -2 \end{bmatrix}$ is 1.

Or in other words, solution space for

$$\begin{bmatrix} -6 & 3 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ is 1-dimensional}$$

Thus a basis for the eigenspace for $\lambda = 7$ is $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$

$\lambda = -1 \quad A - (-1)I = \begin{bmatrix} 1+1 & 3 \\ 4 & 5+1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$

Note $\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Thus a basis for the eigenspace for $\lambda = -1$ is $\left\{ \begin{bmatrix} 3 \\ -2 \end{bmatrix} \right\}$

Thus a basis for the solution space to $\mathbf{X}' = \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} \mathbf{X}$ is

$$\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{7t}, \ \begin{bmatrix} 3 \\ -2 \end{bmatrix} e^{-t} \right\}$$

Hence the general solution is

$$\mathbf{X}(t) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{7t} + c_2 \begin{bmatrix} 3 \\ -2 \end{bmatrix} e^{-t}$$

Note we can take any basis for the solution space to create the general solution

Alternate basis: $\left\{ \begin{bmatrix} 2 \\ 4 \end{bmatrix} e^{7t}, \ \begin{bmatrix} -9 \\ 6 \end{bmatrix} e^{-t} \right\}$

Alternate format of general solution:

$$\mathbf{X}(t) = c_1 \begin{bmatrix} 2 \\ 4 \end{bmatrix} e^{7t} + c_2 \begin{bmatrix} -9 \\ 6 \end{bmatrix} e^{-t}$$

IVP: $\mathbf{X}' = \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} \mathbf{X}, \quad \mathbf{X}(t_0) = \begin{bmatrix} e \\ f \end{bmatrix}$

$$\begin{bmatrix} e \\ f \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{7t_0} + c_2 \begin{bmatrix} 3 \\ -2 \end{bmatrix} e^{-t_0} = \begin{bmatrix} c_1e^{7t_0} + 3c_2e^{-t_0} \\ 2c_1e^{7t_0} - 2c_2e^{-t_0} \end{bmatrix}$$

Solve using any method you like. We will use matrix form:

$$\begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} e^{7t_0} & 3e^{-t_0} \\ 2e^{7t_0} & -2e^{-t_0} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Solution exists if Wronskian evaluated at t_0 is not zero.

$$W\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{7t}, \begin{bmatrix} 3 \\ -2 \end{bmatrix} e^{-t} \right) = \begin{vmatrix} e^{7t} & 3e^{-t} \\ 2e^{7t} & -2e^{-t} \end{vmatrix}$$

$$= -2e^{6t} - 6e^{6t} = -8e^{6t} \neq 0$$

Fundamental matrix: $\Phi(t) = \begin{bmatrix} e^{7t} & 3e^{-t} \\ 2e^{7t} & -2e^{-t} \end{bmatrix}$

Back to IVP: $\begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} e^{7t_0} & 3e^{-t_0} \\ 2e^{7t_0} & -2e^{-t_0} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

$$\begin{bmatrix} e^{7t_0} & 3e^{-t_0} \\ 2e^{7t_0} & -2e^{-t_0} \end{bmatrix}^{-1} \begin{bmatrix} e \\ f \end{bmatrix}$$
$$= \begin{bmatrix} e^{7t_0} & 3e^{-t_0} \\ 2e^{7t_0} & -2e^{-t_0} \end{bmatrix}^{-1} \begin{bmatrix} e^{7t_0} & 3e^{-t_0} \\ 2e^{7t_0} & -2e^{-t_0} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Thus $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} e^{7t_0} & 3e^{-t_0} \\ 2e^{7t_0} & -2e^{-t_0} \end{bmatrix}^{-1} \begin{bmatrix} e \\ f \end{bmatrix}$

But I would prefer a fundamental matrix whose inverse is easier to calculate, at least when $t_0 = 0$.

Thus we will find another basis for the solution set to $\mathbf{x}' = A\mathbf{x}$ so that the corresponding fundamental matrix has the property that $\Phi(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, the 2x2 identity matrix.

Step 1: Solve IVP: $\quad \mathbf{x}' = A\mathbf{x}, \;\; \mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^0 + c_2 \begin{bmatrix} 3 \\ -2 \end{bmatrix} e^0 = \begin{bmatrix} 1 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \text{ implies } \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & -2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$(-\tfrac{1}{8}) \begin{bmatrix} -2 & -3 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ 2 & -2 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{4} & \frac{3}{8} \\ \frac{1}{4} & -\frac{1}{8} \end{bmatrix} \;\&\; \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{3}{8} \\ \frac{1}{4} & -\frac{1}{8} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \end{bmatrix}$$

Thus IVP solution where $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is

$$\mathbf{X}(t) = \tfrac{1}{4} \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{7t} + \tfrac{1}{4} \begin{bmatrix} 3 \\ -2 \end{bmatrix} e^{-t} = \begin{bmatrix} \frac{1}{4}e^{7t} + \frac{3}{4}e^{-t} \\ \frac{1}{2}e^{7t} - \frac{1}{2}e^{-t} \end{bmatrix}$$

Step 2: Solve IVP: $\mathbf{x}' = A\mathbf{x}, \;\; \mathbf{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^0 + c_2 \begin{bmatrix} 3 \\ -2 \end{bmatrix} e^0 = \begin{bmatrix} 1 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Thus $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{3}{8} \\ \frac{1}{4} & -\frac{1}{8} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{8} \\ -\frac{1}{8} \end{bmatrix}$

Thus IVP solution where $\mathbf{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is

$$\mathbf{X}(t) = \tfrac{3}{8} \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{7t} - \tfrac{1}{8} \begin{bmatrix} 3 \\ -2 \end{bmatrix} e^{-t} = \begin{bmatrix} \frac{3}{8}e^{7t} - \frac{3}{8}e^{-t} \\ \frac{3}{4}e^{7t} + \frac{1}{4}e^{-t} \end{bmatrix}$$

Thus another basis for the solution space to $\mathbf{X}' = \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} \mathbf{X}$

$$\text{is} \left\{ \begin{bmatrix} \frac{1}{4}e^{7t} + \frac{3}{4}e^{-t} \\ \frac{1}{2}e^{7t} - \frac{1}{2}e^{-t} \end{bmatrix}, \begin{bmatrix} \frac{3}{8}e^{7t} - \frac{3}{8}e^{-t} \\ \frac{3}{4}e^{7t} + \frac{1}{4}e^{-t} \end{bmatrix} \right\}$$

Its corresponding fundamental matrix is

$$\begin{bmatrix} \frac{1}{4}e^{7t} + \frac{3}{4}e^{-t} & \frac{3}{8}e^{7t} - \frac{3}{8}e^{-t} \\ \frac{1}{2}e^{7t} - \frac{1}{2}e^{-t} & \frac{3}{4}e^{7t} + \frac{1}{4}e^{-t} \end{bmatrix}$$

Thus to solve IVP where $\mathbf{X}(t_0) = \begin{bmatrix} e \\ f \end{bmatrix}$, we solve

$$\begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} \frac{1}{4}e^{7t_0} + \frac{3}{4}e^{-t_0} & \frac{3}{8}e^{7t_0} - \frac{3}{8}e^{-t_0} \\ \frac{1}{2}e^{7t_0} - \frac{1}{2}e^{-t_0} & \frac{3}{4}e^{7t_0} + \frac{1}{4}e^{-t_0} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

When $t_0 = 0$. I.e., we have an IVP where $\mathbf{X}(0) = \begin{bmatrix} e \\ f \end{bmatrix}$

$$\begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} \frac{1}{4}e^{0} + \frac{3}{4}e^{0} & \frac{3}{8}e^{0} - \frac{3}{8}e^{0} \\ \frac{1}{2}e^{0} - \frac{1}{2}e^{0} & \frac{3}{4}e^{0} + \frac{1}{4}e^{0} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

which simplifies to

$$\begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

In other words, $c_1 = e$ and $c_2 = f$.