

Section 3.3: If $b^2 - 4ac < 0$, :

Changed format of $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ to linear combination of real-valued functions instead of complex valued functions by using Euler's formula:

$$e^{it} = \cos(t) + i \sin(t)$$

Hence $e^{(d+in)t} = e^{dt} e^{int} = e^{dt} [\cos(nt) + i \sin(nt)]$

Let $r_1 = d + in$, $r_2 = d - in$

$$\begin{aligned} y &= c_1 e^{r_1 t} + c_2 e^{r_2 t} \\ &= c_1 e^{dt} [\cos(nt) + i \sin(nt)] + c_2 e^{dt} [\cos(-nt) + i \sin(-nt)] \\ &= c_1 e^{dt} \cos(nt) + i c_1 e^{dt} \sin(nt) + c_2 e^{dt} \cos(nt) - i c_2 e^{dt} \sin(nt) \\ &= (c_1 + c_2) e^{dt} \cos(nt) + i (c_1 - c_2) e^{dt} \sin(nt) \\ &= k_1 e^{dt} \cos(nt) + k_2 e^{dt} \sin(nt) \end{aligned}$$

Case 3: $(a + d)^2 - 4(ad - bc) < 0$. I.e., $r = \lambda \pm i\mu$

Suppose the eigenvector corresponding to this eigenvalue is

$$\begin{pmatrix} v_1 + iw_1 \\ v_2 + iw_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + i \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

Then general solution is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} v_1 \cos(\mu t) - w_1 \sin(\mu t) \\ v_2 \cos(\mu t) - w_2 \sin(\mu t) \end{pmatrix} e^{\lambda t} + c_2 \begin{pmatrix} v_1 \sin(\mu t) + w_1 \cos(\mu t) \\ v_2 \sin(\mu t) + w_2 \cos(\mu t) \end{pmatrix} e^{\lambda t}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} v_1 \cos(\mu t) - w_1 \sin(\mu t) \\ v_2 \cos(\mu t) - w_2 \sin(\mu t) \end{pmatrix} e^{\lambda t} + c_2 \begin{pmatrix} v_1 \sin(\mu t) + w_1 \cos(\mu t) \\ v_2 \sin(\mu t) + w_2 \cos(\mu t) \end{pmatrix} e^{\lambda t}$$

Case 3a: $\lambda > 0$

Case 3a: $\lambda < 0$

Case 3a: $\lambda = 0$