Plug in $\vec{x} = \vec{v}e^{rt} \Rightarrow \vec{v}$ is an e. vector w/ e. value r

Section 7.5 (where A has 2 real distinct nonzero eigenvalues): Solve $\mathbf{x}' = \begin{bmatrix} -2 & 0 \\ 21 & 5 \end{bmatrix}\mathbf{x}$

Step 1: Find eigenvalues:

matrix is triangular $\Rightarrow$ e. value are on the diagonal

$$det(A - rI) = \begin{bmatrix} -2-r & 0 \\ 21 & 5-r \end{bmatrix} = (-2-r)(5-r) - (21)(0) = 0$$

$$\Rightarrow r = -2, 5$$

Step 2: For each eigenvalue, find one eigenvector. I.e., find one NONZERO solution to $(A - rI)\mathbf{v} = \mathbf{0}$

$$r = -2: \quad A - (-2I) = A + 2I = \begin{bmatrix} -2+2 & 0 \\ 21 & 5+2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 21 & 7 \end{bmatrix}$$

Solve $(A+2I)\vec{v} = 0$:

$$\begin{bmatrix} 0 & 0 \\ 21 & 7 \end{bmatrix}\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \vec{v} = \begin{bmatrix} -1 \\ 3 \end{bmatrix} \text{ or } \begin{bmatrix} 1 \\ -3 \end{bmatrix} \text{ or } \begin{bmatrix} 7 \\ -21 \end{bmatrix} \text{ or etc}$$

$$r = 5: \quad A - 5I = \begin{bmatrix} -2-5 & 0 \\ 21 & 5-5 \end{bmatrix} = \begin{bmatrix} -7 & 0 \\ 21 & 0 \end{bmatrix}$$

Solve $(A - 5I)\vec{v} = 0$:

$$\begin{bmatrix} -7 & 0 \\ 21 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ or $\begin{bmatrix} 0 \\ 7 \end{bmatrix}$ or ...

General solution: $\vec{x} = c_1\begin{bmatrix} -1 \\ 3 \end{bmatrix}e^{-2t} + c_2\begin{bmatrix} 0 \\ 1 \end{bmatrix}e^{5t}$

special case : $c_1 = 1$, $c_2 = 0$

Initial Value: $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$, IVP solution: $\vec{x} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}e^{-2t}$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1e^{-2t} \\ 3e^{-2t} \end{bmatrix}$$

For this IVP solution:

$x_1(t) =$ $x_1 = -1e^{-2t}$

$x_2(t) =$ $x_2 = 3e^{-2t}$

Solve for $\cancel{x_1}$ x_2 in terms of $\cancel{x_2}$ x_1: divide

$$\frac{x_2}{x_1} = \frac{3\cancel{e^{-2t}}}{-1\cancel{e^{-2t}}} \Rightarrow x_2 = \frac{3}{-1}x_1$$

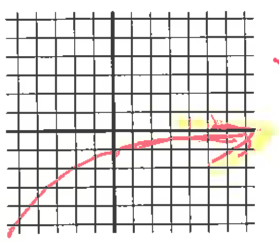
Give that the solution to $\mathbf{x}' = \begin{bmatrix} -2 & 0 \\ 21 & 5 \end{bmatrix} \mathbf{x}$ is $\mathbf{x} = c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{5t} + c_2 \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{-2t}$

$$\frac{X_2}{X_1} = \frac{1e^{5t}}{0e^{5t}} = X_2 = \frac{1}{0} X_1$$

Graph the solution to the IVP $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ in the

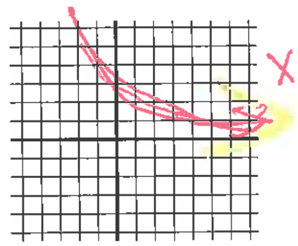
$X_1 = -e^{-2t}$ $\quad X_2 = 3e^{-2t}$ $\quad X_2 = \frac{3}{-1} X_1$

t, x_1-plane



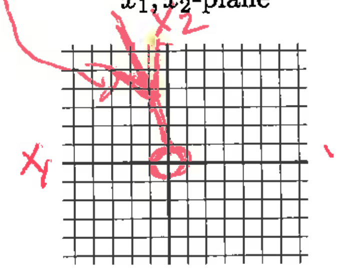
$X_1 < 0$

t, x_2-plane



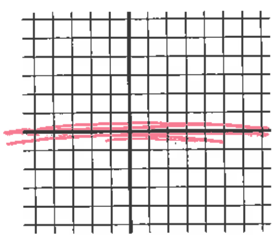
$X_2 > 0$

x_1, x_2-plane

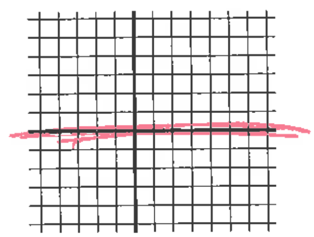


Graph the solution to the IVP $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ in the $\Rightarrow \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

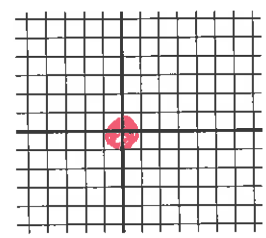
t, x_1-plane



t, x_2-plane



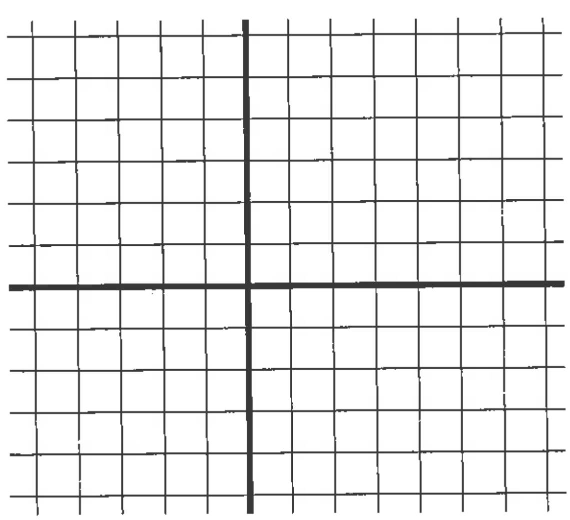
x_1, x_2-plane



The equilibrium solution for this system of equations is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

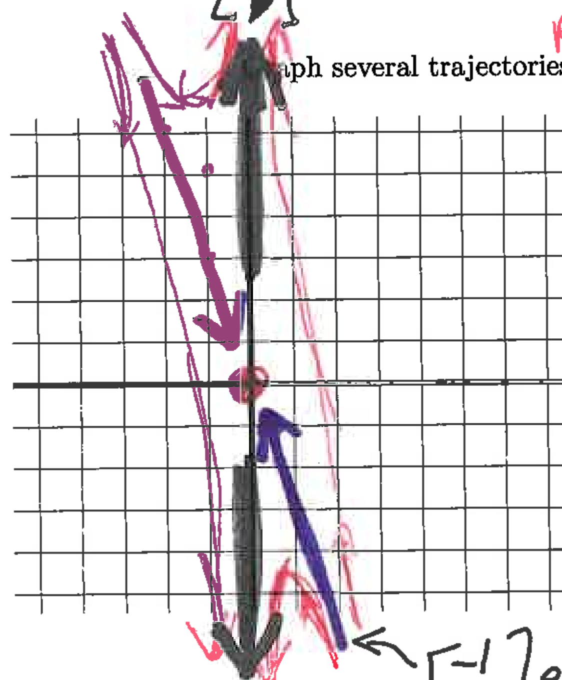
$\frac{dx_2}{dx_1} =$ ____________

Plot several direction vectors where the slope is 0 and where slope is vertical.



$\begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{5t}$ ← dominates for large positive t

ph several trajectories.



$\leftarrow \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{-2t}$

Saddle

Find the **equilibrium solution(s)** for $\mathbf{x}' = A\mathbf{x}$

Recall a solution is an equilibrium solution iff $\mathbf{x}(t) = \mathbf{C}$ iff $\mathbf{x}'(t) = 0$

Setting $\mathbf{x}' = 0$, implies $\mathbf{0} = A\mathbf{x}$.

Thus $\mathbf{x} = \mathbf{C}$ is an equilibrium solution iff it is a solution to $\mathbf{0} = A\mathbf{x}$.

Case 1 (not emphasized/covered): $det(A) = 0$.

In this case, $A\mathbf{x} = \mathbf{0}$ has an infinite number of solutions. Note this case corresponds to the case when 0 is an eigenvalue of A since there are nonzero solutions to $A\mathbf{v} = 0\mathbf{v}$

Case 2: $det(A) \neq 0$.

Then $A\mathbf{x} = \mathbf{0}$ has a unique solution, $\mathbf{x} = \vec{0}$

Thus if $det(A) \neq 0$, $\mathbf{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is the only equilibrium solution of $\mathbf{x}' = A\mathbf{x}$

Slope fields:

* For complex eigenvalue case, one slope is needed.

* For real eigenvalue case, 0 and ∞ slopes can be helpful and can catch graphing errors, but your graph does **not** need to be that accurate.

For $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 21 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} =$

$\frac{dx_1}{dt} =$

$\frac{dx_2}{dt} =$

$\frac{dx_2}{dx_1} =$

Slope 0:

Slope ∞:

e. value 5, −2

Answer the following questions for $A = \begin{bmatrix} -2 & 0 \\ 21 & 5 \end{bmatrix}$:

The smaller eigenvalue of A is $r_1 = $ −2. An eigenvector corresponding to r_1 is $\mathbf{v} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

The larger eigenvalue of A is $r_2 = $ 5. An eigenvector corresponding to r_2 is $\mathbf{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

The general solution to $\mathbf{x}' = A\mathbf{x}$ is

$$\vec{X} = c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{5t} + c_2 \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{-2t}$$ or switch c_1, c_2 just be consistent

---

For large **positive** values of t which is larger: $e^{r_1 t}$ or $e^{r_2 t}$? $e^{-2t} < e^{5t}$

For the following problems, consider the case when $c_1 \neq 0$ and $c_2 \neq 0$ where the general solution is $\mathbf{x} = c_1 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} e^{r_1 t} + c_2 \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} e^{r_2 t}$,

$c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{5t}$

For large **positive** values of t, which term dominates: $c_1 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} e^{r_1 t}$ or $c_2 \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} e^{r_2 t}$?

Thus for large **positive** values of t, such trajectories (where $c_1 c_2 \neq 0$) when projected into the x_1, x_2 plane exhibit the following behavior (select all that apply):

* moves away from the origin. positive exp
* moves toward the origin.
* approaches the line $y = mx$ with slope $m = $ 1/0
* approaches a line $y = mx + b$ for $b \neq 0$ with slope $m = $ ________. Note this case corresponds to where both $||c_1\mathbf{v}||e^{r_1 t}$ and $||c_2\mathbf{w}||e^{r_2 t}$ are large, but one is significantly larger than the other.

$e^{-2t} > e^{5t}$

$c_2 \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{-2t}$

For large **negative** values of t which is larger: $e^{r_1 t}$ or $e^{r_2 t}$?

For large **negative** values of t, which term dominates: $c_1 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} e^{r_1 t}$ or $c_2 \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} e^{r_2 t}$?

Thus for large **negative** values of t, such trajectories (where $c_1 c_2 \neq 0$) when projected into the x_1, x_2 plane exhibit the following behavior (select all that apply):

* moves away from the origin.
* moves toward the origin. negative exp
* approaches the line $y = mx$ with slope $m = $ 3/−1
* approaches a line $y = mx + b$ for $b \neq 0$ with slope $m = $ ________. Note this case corresponds to where both $||c_1\mathbf{v}||e^{r_1 t}$ and $||c_2\mathbf{w}||e^{r_2 t}$ are large, but one is significantly larger than the other.

Give that the solution to $\mathbf{x}' = \begin{bmatrix} -2 & 0 \\ -9 & -5 \end{bmatrix} \mathbf{x}$ is $\mathbf{x} = c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-5t} + c_2 \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{-2t}$

$\frac{X_2}{X_1} = \frac{3}{-1}$

$X_2 = \frac{3}{-1} X_1$

$\frac{X_2}{X_1} = \frac{1}{0} \Rightarrow X_2 = \frac{1}{0} X_1$

Graph the solution to the IVP $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ in the

t, x_1-plane

t, x_2-plane

x_1, x_2-plane

Graph the solution to the IVP $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ in the

t, x_1-plane

t, x_2-plane

x_1, x_2-plane

The equilibrium solution for this system of equations is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

negative exponential
$\Rightarrow$ all arrows point in

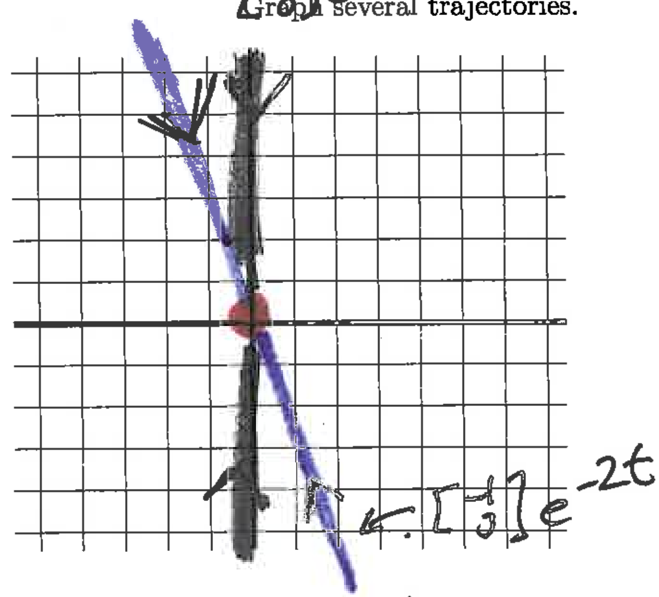
$\frac{dx_2}{dx_1} =$ ____________

Plot several direction vectors where the slope is 0 and where slope is vertical.

$\begin{bmatrix} -1 \\ 3 \end{bmatrix} \rightarrow$ slope $\frac{3}{-1}$

$\begin{bmatrix} -1 \\ 0 \end{bmatrix} e^{-5t}$

Graph several trajectories.



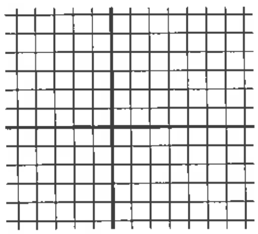
$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightarrow \frac{1}{0}$ slope

Give that the solution to $\mathbf{x}' = \begin{bmatrix} 2 & 0 \\ 9 & 5 \end{bmatrix} \mathbf{x}$ is $\mathbf{x} = c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{5t} + c_2 \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{2t}$

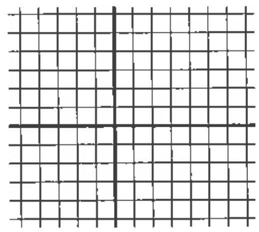
$X_2 = \frac{1}{0} X_1$ $\qquad$ $X_2 = \frac{3}{-1} X_1$

Graph the solution to the IVP $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ in the

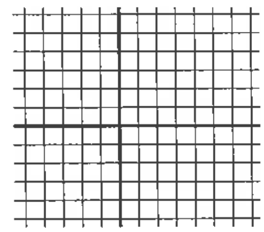
t, x_1-plane



t, x_2-plane

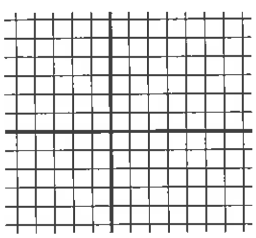


x_1, x_2-plane

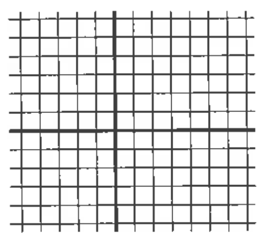


Graph the solution to the IVP $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ in the

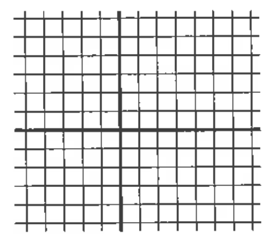
t, x_1-plane



t, x_2-plane



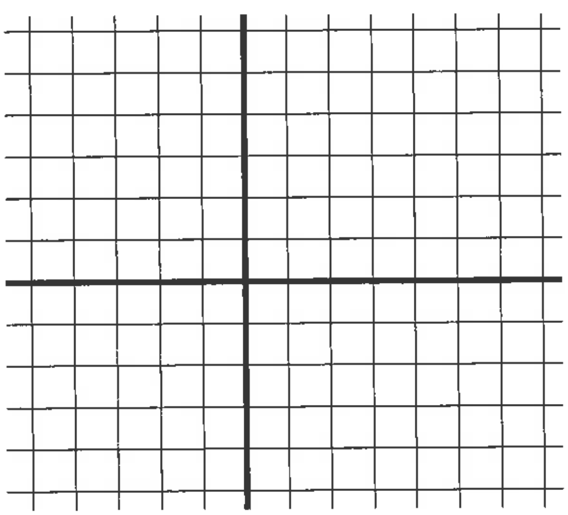
x_1, x_2-plane



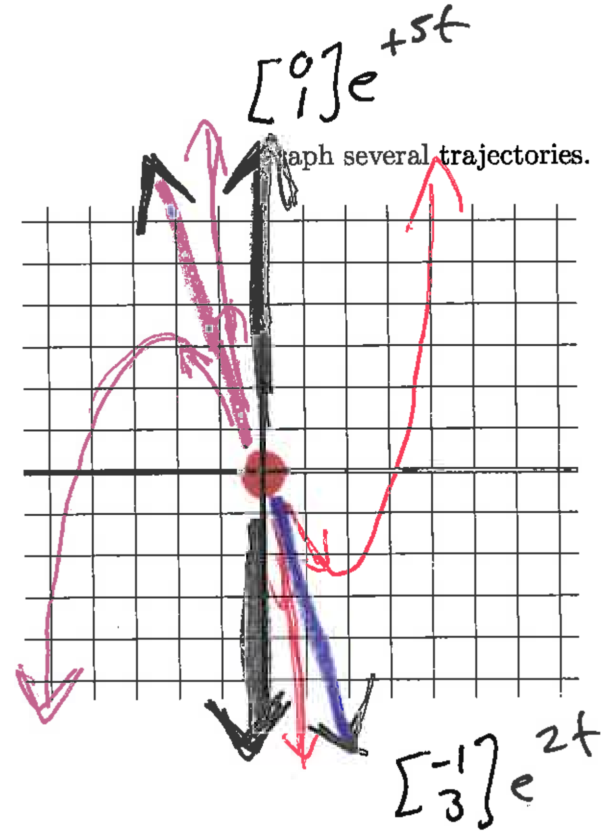
The equilibrium solution for this system of equations is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

$\frac{dx_2}{dx_1} =$ ____________

Plot several direction vectors where the slope is 0 and where slope is vertical.



Graph several trajectories.



Answer the following questions for $A = \begin{bmatrix} 2 & 0 \\ 9 & 5 \end{bmatrix}$:

The smaller eigenvalue of A is $r_1 =$ 2. An eigenvector corresponding to r_1 is $\mathbf{v} =$ $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$

The larger eigenvalue of A is $r_2 =$ 5. An eigenvector corresponding to r_2 is $\mathbf{w} =$ $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

The general solution to $\mathbf{x}' = A\mathbf{x}$ is

For large **positive** values of t which is larger: $e^{r_1 t}$ or $e^{r_2 t}$? $e^{2t} < e^{5t}$

For the following problems, consider the case when $c_1 \neq 0$ and $c_2 \neq 0$ where the general solution is $\mathbf{x} = c_1 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} e^{r_1 t} + c_2 \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} e^{r_2 t}$.

$\begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{5t}$

For large **positive** values of t, which term dominates: $c_1 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} e^{r_1 t}$ or $c_2 \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} e^{r_2 t}$?

Thus for large **positive** values of t, such trajectories (where $c_1 c_2 \neq 0$) when projected into the x_1, x_2 plane exhibit the following behavior (select all that apply):

* moves away from the origin. positive exp
* moves toward the origin.
* approaches the line $y = mx$ with slope $m =$ ________
* approaches a line $y = mx + b$ for $b \neq 0$ with slope $m =$ 1/0. Note this case corresponds to where both $||c_1\mathbf{v}||e^{r_1 t}$ and $||c_2\mathbf{w}||e^{r_2 t}$ are large, but one is significantly larger than the other.

For large **negative** values of t which is larger: $e^{r_1 t}$ or $e^{r_2 t}$? $e^{2t} > e^{5t}$

For large **negative** values of t, which term dominates: $c_1 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} e^{r_1 t}$ or $c_2 \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} e^{r_2 t}$? $\begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{2t}$

Thus for large **negative** values of t, such trajectories (where $c_1 c_2 \neq 0$) when projected into the x_1, x_2 plane exhibit the following behavior (select all that apply):

* moves away from the origin. positive exp
* moves toward the origin.
* approaches the line $y = mx$ with slope $m =$ 3/-1
* approaches a line $y = mx + b$ for $b \neq 0$ with slope $m =$ ________. Note this case corresponds to where both $||c_1\mathbf{v}||e^{r_1 t}$ and $||c_2\mathbf{w}||e^{r_2 t}$ are large, but one is significantly larger than the other.

Solutions: $\vec{x} = \vec{v}e^{rt}$ where $\vec{v}$ is an e.vector w/ e.value r

Section 7.5 (where A has 2 real distinct nonzero eigenvalues): Solve $\mathbf{x}' = \begin{bmatrix} -2 & 0 \\ -9 & -5 \end{bmatrix}\mathbf{x}$

Step 1: Find eigenvalues: matrix is triangular $\Rightarrow$ e.values are on the diagonal

$det(A - rI) = \begin{bmatrix} -2-r & 0 \\ -9 & -5-r \end{bmatrix} = (-2-r)(-5-r) - (-9)(0) = 0$

$\Rightarrow r = -2, -5$

Step 2: For each eigenvalue, find one eigenvector. I.e., find one NONZERO solution to $(A - rI)\mathbf{v} = \mathbf{0}$

$r = -2: \quad A - (-2I) = A + 2I = \begin{bmatrix} -2+2 & 0 \\ -9 & -5+2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -9 & -3 \end{bmatrix}$

Solve $(A+2I)\vec{v} = \vec{0}$: $\begin{bmatrix} 0 & 0 \\ -9 & -3 \end{bmatrix}\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow \vec{v} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ or $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$ or $\begin{bmatrix} -3 \\ +9 \end{bmatrix}$ or $\begin{bmatrix} +3 \\ -9 \end{bmatrix}$ or ....

$r = -5: \quad A - (-5I) = A + 5I = \begin{bmatrix} -2+5 & 0 \\ -9 & -5+5 \end{bmatrix} = \begin{bmatrix} +3 & 0 \\ 9 & 0 \end{bmatrix}$

Solve $(A+5I)\vec{v} = 0$: $\begin{bmatrix} +3 & 0 \\ 9 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\vec{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ or $\begin{bmatrix} 0 \\ -3 \end{bmatrix}$ or ...

General solution: $\vec{x} = c_1\begin{bmatrix} -1 \\ 3 \end{bmatrix}e^{-2t} + c_2\begin{bmatrix} 0 \\ 1 \end{bmatrix}e^{-5t}$

special case ($c_1 = 1$, $c_2 = 0$)

Initial Value: $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$, IVP solution: $\vec{x} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}e^{-2t}$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1e^{-2t} \\ 3e^{-2t} \end{bmatrix}$

For this IVP solution:

$x_1(t) = -1e^{-2t}$

$x_2(t) = 3e^{-2t}$

Solve for x_2 in terms of x_1: divide

$\frac{x_2}{x_1} = \frac{3\cancel{e^{-2t}}}{-1\cancel{e^{-2t}}} = \quad x_2 = \frac{3}{-1}x_1$

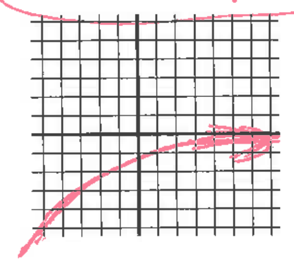
Give that the solution to $\mathbf{x}' = \begin{bmatrix}-2 & 0\\ -9 & -5\end{bmatrix}\mathbf{x}$ is $\mathbf{x} = c_1\begin{bmatrix}0\\ 1\end{bmatrix}e^{-5t} + c_2\begin{bmatrix}-1\\ 3\end{bmatrix}e^{-2t}$

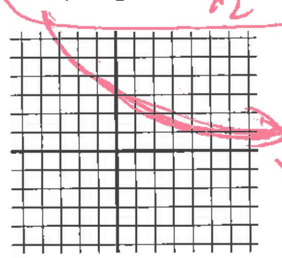
$\frac{x_2}{x_1} = \frac{1e^{-5t}}{0e^{-5t}} \Rightarrow x_2 = \frac{1}{0}x_1$ $\qquad \begin{bmatrix}-1\\ 3\end{bmatrix}e^{-2t}$

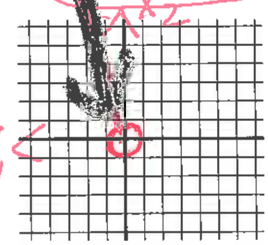
Graph the solution to the IVP $\begin{bmatrix}x_1(0)\\ x_2(0)\end{bmatrix} = \begin{bmatrix}-1\\ 3\end{bmatrix}$ in the

$x_2 = \frac{3}{-1}x_1$

t, x_1-plane $x_1 = -e^{-2t}$ $\qquad$ t, x_2-plane $x_2 = 3e^{-2t}$ $\qquad$ x_1, x_2-plane

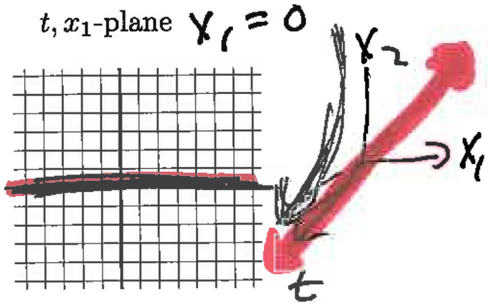


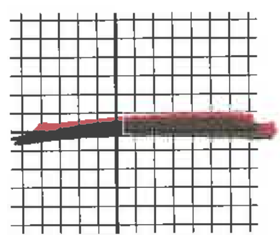


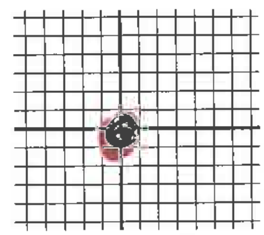


Graph the solution to the IVP $\begin{bmatrix}x_1(0)\\ x_2(0)\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$ in the $\quad c_1 = c_2 = 0 \Rightarrow x_1(t) = 0$ & $x_2(t) = 0$

t, x_1-plane $x_1 = 0$ $\qquad$ t, x_2-plane $x_2 = 0$ $\qquad$ x_1, x_2-plane







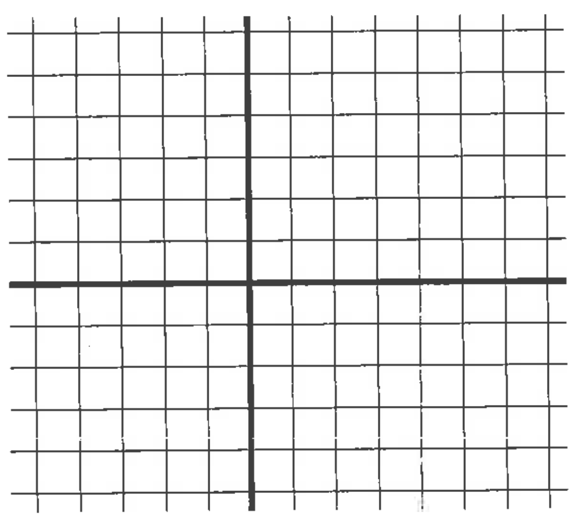
The equilibrium solution for this system of equations is $\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$.

dominates for large positive t $\qquad c_2\begin{bmatrix}-1\\ 3\end{bmatrix}e^{-2t}, c_2 > 0$

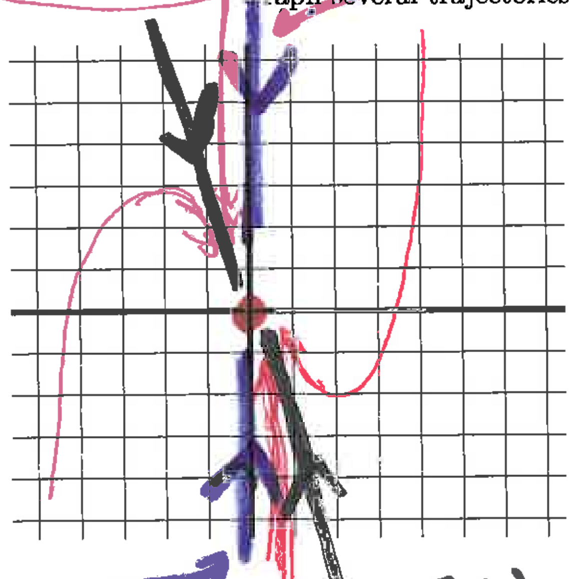
dominates for neg value of t $\qquad c_1\begin{bmatrix}0\\ 1\end{bmatrix}e^{-5t}$

$\frac{dx_2}{dx_1} =$ ____________

Plot several direction vectors where the slope is 0 and where slope is vertical.



Graph several trajectories.



slope $\frac{1}{0}$

neg exp

$c_1\begin{bmatrix}0\\ 1\end{bmatrix}e^{-5t}$ $\qquad$ $c_2\begin{bmatrix}-1\\ 3\end{bmatrix}e^{-2t}, c_2 < 0$

Answer the following questions for $A = \begin{bmatrix} -2 & 0 \\ -9 & -5 \end{bmatrix}$: $\Rightarrow r = -5, -2$

The smaller eigenvalue of A is $r_1 =$ -5. An eigenvector corresponding to r_1 is $\mathbf{v} =$ $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

The larger eigenvalue of A is $r_2 =$ -2. An eigenvector corresponding to r_2 is $\mathbf{w} =$ $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$

The general solution to $\mathbf{x}' = A\mathbf{x}$ is going backwards in time

$$\vec{X} = c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-5t} + c_2 \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{-2t}$$

For large **positive** values of t which is larger: $e^{r_1 t}$ or $e^{r_2 t}$? $e^{-5t} < e^{-2t}$

For the following problems, consider the case when $c_1 \neq 0$ and $c_2 \neq 0$ where the general solution is $\mathbf{x} = c_1 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} e^{r_1 t} + c_2 \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} e^{r_2 t}$,

$\begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{-2t}$

For large **positive** values of t, which term dominates: $c_1 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} e^{r_1 t}$ or $c_2 \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} e^{r_2 t}$?

Thus for large **positive** values of t, such trajectories (where $c_1 c_2 \neq 0$) when projected into the x_1, x_2 plane exhibit the following behavior (select all that apply):

* moves away from the origin.
* moves toward the origin. ← negative exponential
* approaches the line $y = mx$ with slope $m =$ $3/-1$ ← since $\begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{-2t}$ term dominates
* approaches a line $y = mx + b$ for $b \neq 0$ with slope $m =$ ______. Note this case corresponds to where both $||c_1\mathbf{v}||e^{r_1 t}$ and $||c_2\mathbf{w}||e^{r_2 t}$ are large, but one is significantly larger than the other.

For large **negative** values of t which is larger: $e^{r_1 t}$ or $e^{r_2 t}$? $e^{-5t} > e^{-2t}$

For large **negative** values of t, which term dominates: $c_1 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} e^{r_1 t}$ or $c_2 \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} e^{r_2 t}$?

Thus for large **negative** values of t, such trajectories (where $c_1 c_2 \neq 0$) when projected into the x_1, x_2 plane exhibit the following behavior (select all that apply):

$\begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-5t}$ ← dominate for negative t-values

* going backward in time — moves away from the origin.
* moves toward the origin. ← negative exp going forward in time
* approaches the line $y = mx$ with slope $m =$ $1/0$
* approaches a line $y = mx + b$ for $b \neq 0$ with slope $m =$ ______. Note this case corresponds to where both $||c_1\mathbf{v}||e^{r_1 t}$ and $||c_2\mathbf{w}||e^{r_2 t}$ are large, but one is significantly larger than the other.

Find the **equilibrium solution(s)** for $\mathbf{x}' = A\mathbf{x}$

Recall a solution is an equilibrium solution iff $\boxed{\mathbf{x}(t) = \mathbf{C}}$ iff $\mathbf{x}'(t) = 0$

Setting $\mathbf{x}' = 0$, implies $\mathbf{0} = A\mathbf{x}$.

Thus $\mathbf{x} = \mathbf{C}$ is an equilibrium solution iff it is a solution to $\mathbf{0} = A\mathbf{x}$.

Case 1 (not emphasized/covered): $det(A) = 0$.

In this case, $A\mathbf{x} = \mathbf{0}$ has an infinite number of solutions. Note this case corresponds to the case when 0 is an eigenvalue of A since there are nonzero solutions to $A\mathbf{v} = 0\mathbf{v}$

Case 2: $det(A) \neq 0$.

Then $A\mathbf{x} = \mathbf{0}$ has a unique solution, $\mathbf{x} =$ $\vec{0}$

Thus if $det(A) \neq 0$, $\mathbf{x} =$ $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is the only equilibrium solution of $\mathbf{x}' = A\mathbf{x}$

Slope fields:

* For complex eigenvalue case, one slope is needed.

* For real eigenvalue case, 0 and ∞ slopes can be helpful and can catch graphing errors, but your graph does **not** need to be that accurate.

For $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 21 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} =$

$\frac{dx_1}{dt} =$

$\frac{dx_2}{dt} =$

$\frac{dx_2}{dx_1} =$

Slope 0:

Slope ∞:

Give that the solution to $\mathbf{x}' = \begin{bmatrix} -2 & 0 \\ 21 & 5 \end{bmatrix} \mathbf{x}$ is $\mathbf{x} = c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{5t} + c_2 \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{-2t}$

$\frac{x_2}{x_1} = \frac{1}{0} \Rightarrow x_2 = \frac{1}{0} x_1$

$\frac{x_2}{x_1} = \frac{3}{-1} \Rightarrow x_2 = \frac{3}{-1} x$

Graph the solution to the IVP $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ in the

t, x_1-plane

t, x_2-plane

x_1, x_2-plane

Graph the solution to the IVP $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ in the

t, x_1-plane

t, x_2-plane

x_1, x_2-plane

The equilibrium solution for this system of equations is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

$\begin{bmatrix} -1 \\ 3 \end{bmatrix} \to$ slope $\frac{3}{-1}$

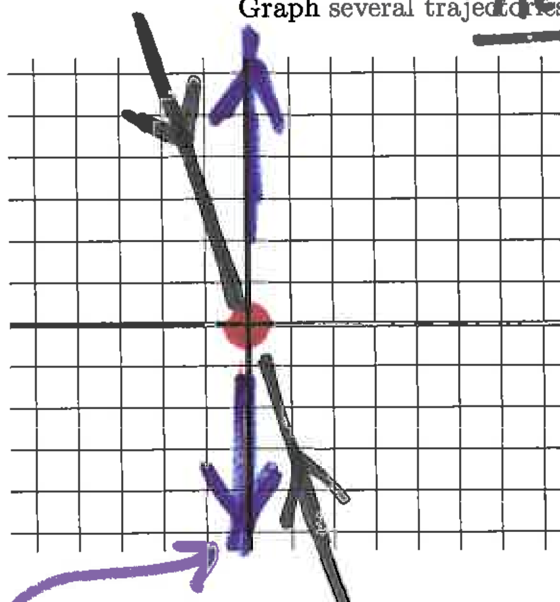
$\begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{-2t}$

neg exp

$\frac{dx_2}{dx_1} =$ ____________

Plot several direction vectors where the slope is 0 and where slope is vertical.

Graph several trajectories.



$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \to$ slope $\frac{1}{0}$

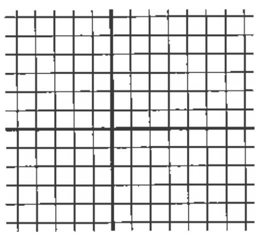
$\begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{5t}$

pos exp

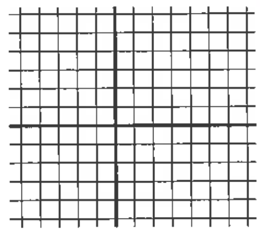
Give that the solution to $\mathbf{x}' = \begin{bmatrix} 2 & 0 \\ 9 & 5 \end{bmatrix} \mathbf{x}$ is $\mathbf{x} = c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{5t} + c_2 \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{2t}$

Graph the solution to the IVP $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ in the

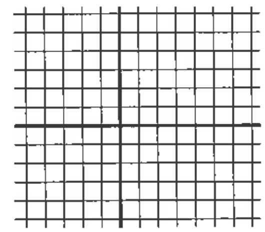
t, x_1-plane



t, x_2-plane

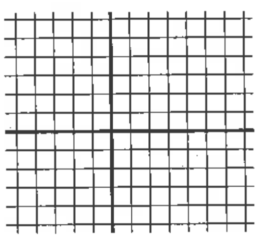


x_1, x_2-plane

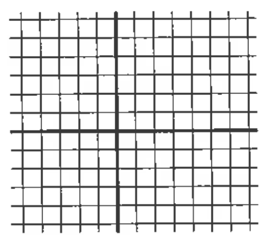


Graph the solution to the IVP $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ in the

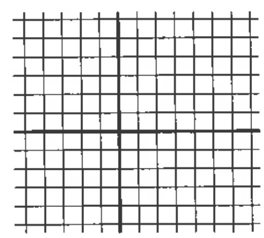
t, x_1-plane



t, x_2-plane



x_1, x_2-plane

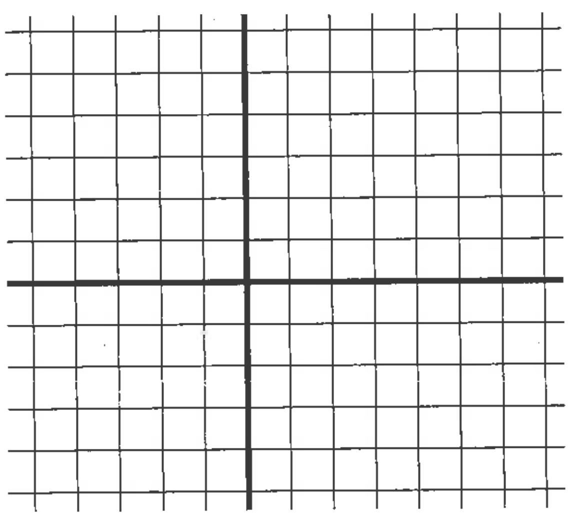


The equilibrium solution for this system of equations is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

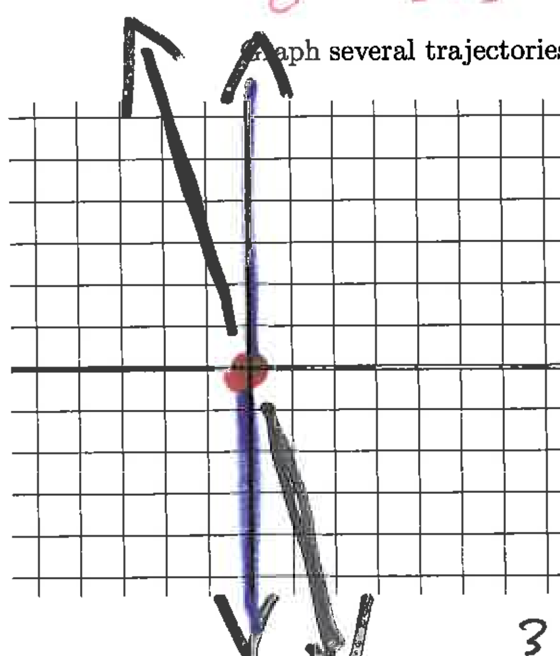
$\frac{1}{0} \leftarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{5t}$

$\frac{dx_2}{dx_1} =$ ______________

Plot several direction vectors where the slope is 0 and where slope is vertical.



Graph several trajectories.



positive exp

$\frac{3}{-1} \leftarrow \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{t}$

$\vec{x}' = A\vec{x} \Rightarrow \vec{x} = \vec{v}e^{rt}$ where $\vec{v}$ is an e. vector w/ e. value r

Section 7.5 (where A has 2 real distinct nonzero eigenvalues): Solve $\mathbf{x}' = \begin{bmatrix} 2 & 0 \\ 9 & 5 \end{bmatrix}\mathbf{x}$

Step 1: Find eigenvalues: Triangular matrix $\Rightarrow$ e. values are along the diagonal

$det(A - rI) = \begin{bmatrix} 2-r & 0 \\ 9 & 5-r \end{bmatrix} = (2-r)(5-r) - 9(0) = 0$

$\Rightarrow r = 2, 5$

Step 2: For each eigenvalue, find one eigenvector. I.e., find one NONZERO solution to $(A - rI)\mathbf{v} = \mathbf{0}$

$r = 2: \quad A - 2I = \begin{bmatrix} 2-2 & 0 \\ 9 & 5-2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 9 & 3 \end{bmatrix}$

Solve $(A - 2I)\vec{v} = 0$: $\begin{bmatrix} 0 & 0 \\ 9 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow \vec{v} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ or $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$ or $\begin{bmatrix} -3 \\ 9 \end{bmatrix}$ or $\begin{bmatrix} 3 \\ -9 \end{bmatrix}$ or ...

$r = 5: A - 5I = \begin{bmatrix} 2-5 & 0 \\ 9 & 5-5 \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 9 & 0 \end{bmatrix}$

Solve $(A - 5I)\vec{v} = 0 \Rightarrow \begin{bmatrix} -3 & 0 \\ 9 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow \vec{v} = \begin{bmatrix} 0 \\ +3 \end{bmatrix}$ or $\begin{bmatrix} -0 \\ 9 \end{bmatrix}$ or $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

2 eqns (1st order linear) $\Rightarrow$ 2 constants

General solution: $\vec{x} = c_1\begin{bmatrix} -1 \\ 3 \end{bmatrix}e^{2t} + c_2\begin{bmatrix} 0 \\ 1 \end{bmatrix}e^{5t}$

Special case $c_1 = 1, c_2 = 0$

$\rightarrow$ Initial Value: $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$, IVP solution: $\vec{x} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}e^{2t}$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -e^{2t} \\ 3e^{2t} \end{bmatrix}$

For this IVP solution:

$x_1(t) =$ $x_1 = -e^{2t}$

$x_2(t) =$ $x_2 = 3e^{2t}$

Solve for x_2 in terms of x_1: divide

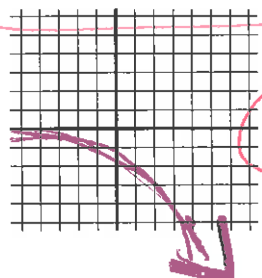
$$\frac{x_2}{x_1} = \frac{3e^{2t}}{-e^{2t}} = \frac{3\cancel{e^{2t}}}{-\cancel{e^{2t}}} = \frac{3}{-1} \Rightarrow x_2 = \frac{3}{-1}x_1$$

Give that the solution to $\mathbf{x}' = \begin{bmatrix}2 & 0\\ 9 & 5\end{bmatrix}\mathbf{x}$ is $\mathbf{x} = c_1\begin{bmatrix}0\\ 1\end{bmatrix}e^{5t} + c_2\begin{bmatrix}-1\\ 3\end{bmatrix}e^{2t}$

$$\frac{x_2}{x_1} = \frac{1c_1e^{5t}}{0c_1e^{5t}} = \frac{1}{0} \Rightarrow x_2 = \frac{1}{0}x_1$$

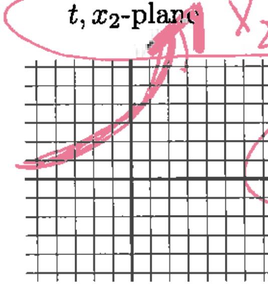
Graph the solution to the IVP $\begin{bmatrix}x_1(0)\\ x_2(0)\end{bmatrix} = \begin{bmatrix}-1\\ 3\end{bmatrix}$ in the

t, x_1-plane $x_1 = -e^{2t}$



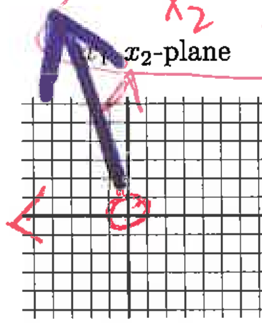
$x_1 < 0$

t, x_2-plane $x_2 = 3e^{2t}$



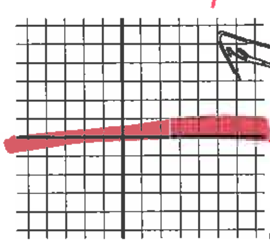
$x_2 > 0$

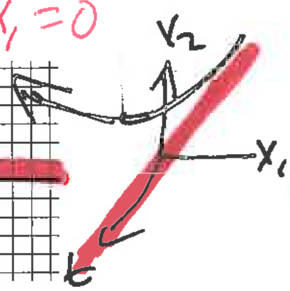
x_1, x_2-plane $x_2 = \frac{3}{-1}x_1$



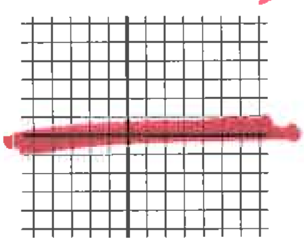
Graph the solution to the IVP $\begin{bmatrix}x_1(0)\\ x_2(0)\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$ in the $\Rightarrow c_1 = c_2 = 0 \Rightarrow x_1(t) = 0$, $x_2(t) = 0$

t, x_1-plane $x_1 = 0$

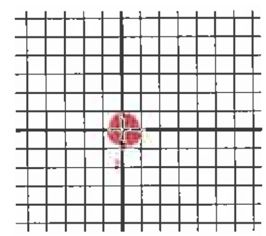




t, x_2-plane $x_2 = 0$



x_1, x_2-plane

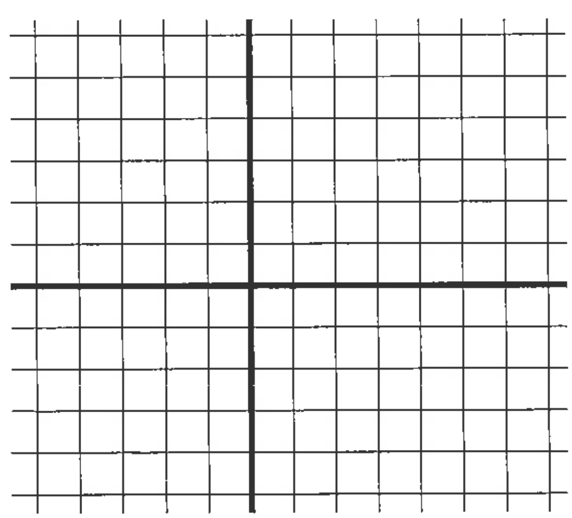


The equilibrium solution for this system of equations is $\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$.

$x = c_2\begin{bmatrix}-1\\ 3\end{bmatrix}e^{2t}$, $c_1 > 0$

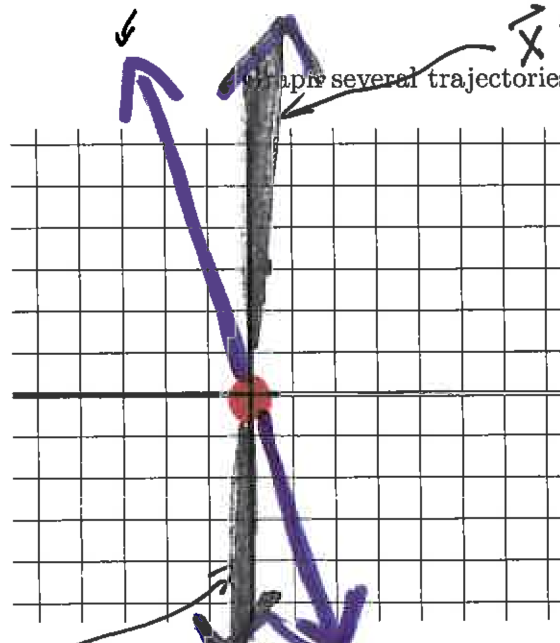
$\frac{dx_2}{dx_1} =$ ______

Plot several direction vectors where the slope is 0 and where slope is vertical.



Graph several trajectories.

$\vec{x} = c_1\begin{bmatrix}0\\ 1\end{bmatrix}e$



positive e.value thus solns go away from 0

$\frac{0}{1} \Rightarrow$ slope $\frac{1}{0}$

$\vec{x} = c_1\begin{bmatrix}0\\ 1\end{bmatrix}e^{5t}$

$\vec{x} = c_2\begin{bmatrix}-1\\ 3\end{bmatrix}e^{2t}$, $c_2 < 0$

$c_1 = c_2 = 0 \Rightarrow \vec{X} = 0$

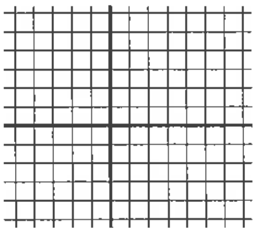
Give that the solution to $\mathbf{x}' = \begin{bmatrix} -2 & 0 \\ 21 & 5 \end{bmatrix} \mathbf{x}$ is $\mathbf{x} = c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{5t} + c_2 \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{-2t}$

$\frac{X_2}{X_1} = \frac{1}{0} \Rightarrow X_2 = \frac{1}{0} X_1$

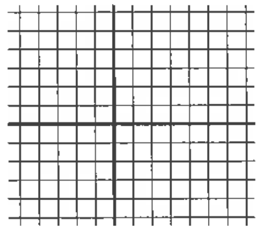
$\frac{X_2}{X_1} = \frac{3}{-1} \Rightarrow X_2 = \frac{3}{-1} X_1$

Graph the solution to the IVP $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ in the

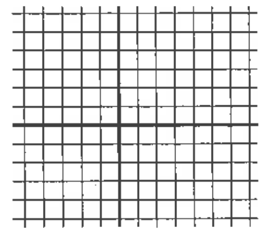
t, x_1-plane



t, x_2-plane

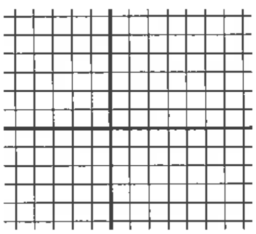


x_1, x_2-plane

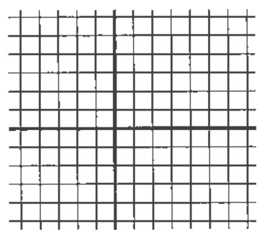


Graph the solution to the IVP $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ in the

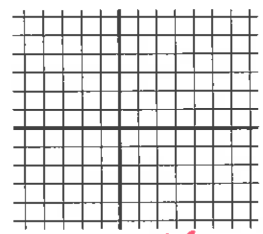
t, x_1-plane



t, x_2-plane



x_1, x_2-plane



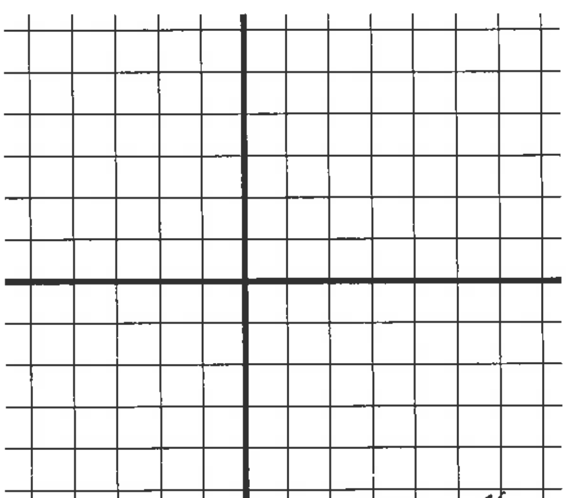
The equilibrium solution for this system of equations is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

$\begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{5t} \Leftarrow$ positive e-value goes away from

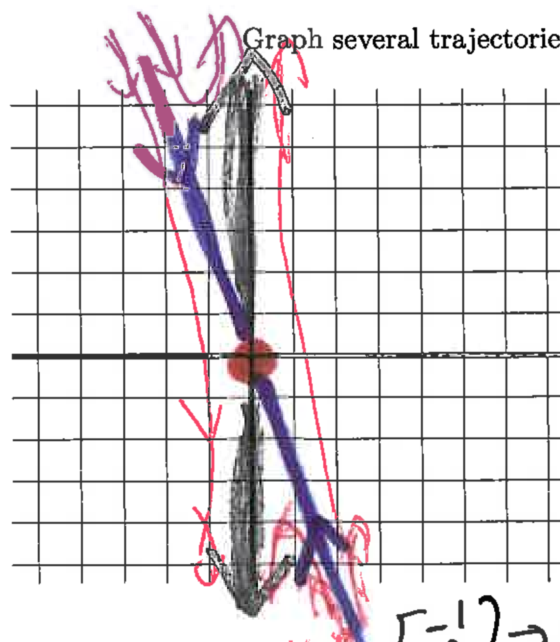
$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \to$ slope $\frac{1}{0}$

$\frac{dx_2}{dx_1} =$ ______________

Plot several direction vectors where
the slope is 0 and where slope is vertical.



Graph several trajectories.



negative e-value $\Rightarrow$ goes to 0

$\begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{-2t}$

$\begin{bmatrix} -1 \\ 3 \end{bmatrix} \to$ slope $\frac{3}{-1}$

$t \to +\infty \Rightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{5t}$ dominant

as $t \to -\infty \Rightarrow \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{-2t}$ dominate

Saddle

Give that the solution to $\mathbf{x}' = \begin{bmatrix} -2 & 0 \\ -9 & -5 \end{bmatrix} \mathbf{x}$ is $\mathbf{x} = c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-5t} + c_2 \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{-2t}$

Graph the solution to the IVP $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ in the

t, x_1-plane

t, x_2-plane

x_1, x_2-plane

Graph the solution to the IVP $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ in the

t, x_1-plane

t, x_2-plane

x_1, x_2-plane

The equilibrium solution for this system of equations is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

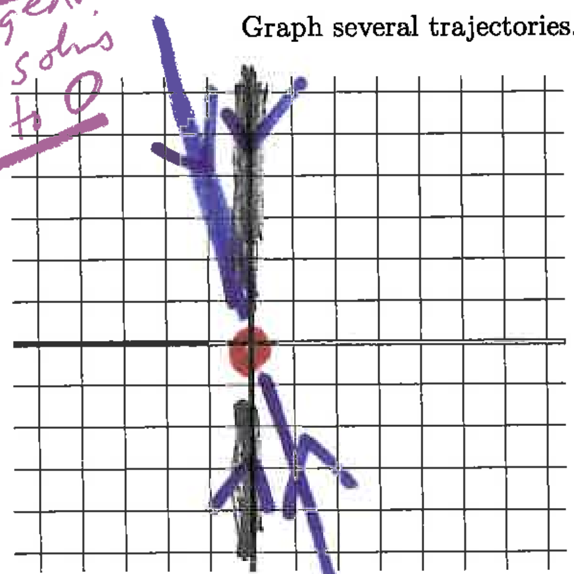
Slope $\frac{1}{0}$

$\frac{dx_2}{dx_1} =$ ____________

negative eigenvalue $\Rightarrow$ solns go to 0

Plot several direction vectors where the slope is 0 and where slope is vertical.

Graph several trajectories.



Slope $\frac{3}{-1}$

constant soln

Find the **equilibrium solution(s)** for $\mathbf{x}' = A\mathbf{x}$

Recall a solution is an equilibrium solution iff $\boxed{\mathbf{x}(t) = \mathbf{C}}$ iff $\mathbf{x}'(t) = 0$

Setting $\mathbf{x}' = 0$, implies $\mathbf{0} = A\mathbf{x}$.

Thus $\mathbf{x} = \mathbf{C}$ is an equilibrium solution iff it is a solution to $\mathbf{0} = A\mathbf{x}$.

Case 1 (not emphasized/covered): $det(A) = 0$.

In this case, $A\mathbf{x} = \mathbf{0}$ has an infinite number of solutions. Note this case corresponds to the case when 0 is an eigenvalue of A since there are nonzero solutions to $A\mathbf{v} = 0\mathbf{v}$

Case 2: $det(A) \neq 0$.

Then $A\mathbf{x} = \mathbf{0}$ has a unique solution, $\mathbf{x} = \vec{0}$

Thus if $det(A) \neq 0$, $\mathbf{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is the only equilibrium solution of $\mathbf{x}' = A\mathbf{x}$

Slope fields:

* For complex eigenvalue case, one slope is needed.

* For real eigenvalue case, 0 and ∞ slopes can be helpful and can catch graphing errors, but your graph does **not** need to be that accurate.

For $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 21 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} =$

$\frac{dx_1}{dt} =$

$\frac{dx_2}{dt} =$

$\frac{dx_2}{dx_1} =$

Slope 0:

Slope ∞:

Answer the following questions for $A = \begin{bmatrix} -2 & 0 \\ 21 & 5 \end{bmatrix}$:

The smaller eigenvalue of A is $r_1 =$ -2. An eigenvector corresponding to r_1 is $\mathbf{v} =$ $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$

The larger eigenvalue of A is $r_2 =$ 5. An eigenvector corresponding to r_2 is $\mathbf{w} =$ $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

The general solution to $\mathbf{x}' = A\mathbf{x}$ is

$$\vec{x} = c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{5t} + c_2 \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{-2t}$$

For large **positive** values of t which is larger: $e^{r_1 t}$ or $e^{r_2 t}$? $e^{-2t} < e^{5t}$ Suppose neither c_1 nor $c_2 = 0$

For the following problems, consider the case when $c_1 \neq 0$ and $c_2 \neq 0$ where the general solution is $\mathbf{x} = c_1 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} e^{r_1 t} + c_2 \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} e^{r_2 t}$,

$\begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{5t}$

For large **positive** values of t, which term dominates: $c_1 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} e^{r_1 t}$ or $c_2 \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} e^{r_2 t}$? larger for positive t

Thus for large **positive** values of t, such trajectories (where $c_1 c_2 \neq 0$) when projected into the x_1, x_2 plane exhibit the following behavior (select all that apply):

* moves away from the origin. positive evalu $\begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{5t}$
* moves toward the origin.
* approaches the line $y = mx$ with slope $m =$ $\frac{1}{0}$
* approaches a line $y = mx + b$ for $b \neq 0$ with slope $m =$ ______. Note this case corresponds to where both $||c_1 \mathbf{v}||e^{r_1 t}$ and $||c_2 \mathbf{w}||e^{r_2 t}$ are large, but one is significantly larger than the other.

For large **negative** values of t which is larger: $e^{r_1 t}$ or $e^{r_2 t}$? $e^{5t} < e^{-2t}$

For large **negative** values of t, which term dominates: $c_1 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} e^{r_1 t}$ or $c_2 \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} e^{r_2 t}$?

Thus for large **negative** values of t, such trajectories (where $c_1 c_2 \neq 0$) when projected into the x_1, x_2 plane exhibit the following behavior (select all that apply):

$\begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{-2t}$

* moves away from the origin.
* moves toward the origin.
* approaches the line $y = mx$ with slope $m =$ $3/-1$
* approaches a line $y = mx + b$ for $b \neq 0$ with slope $m =$ ______. Note this case corresponds to where both $||c_1 \mathbf{v}||e^{r_1 t}$ and $||c_2 \mathbf{w}||e^{r_2 t}$ are large, but one is significantly larger than the other.