7.4 - 7.6, 9.1

Solve the homogeneous linear DE: $\mathbf{x}' - A\mathbf{x} = \mathbf{0}$

$\mathbf{x}' = A\mathbf{x}$ Guess $x = \mathbf{v}e^{rt}$. Plug in to find $\mathbf{v}$ and r:

$$[\mathbf{v}e^{rt}]' = A\mathbf{v}e^{rt} \quad \text{implies} \quad r\mathbf{v}e^{rt} = A\mathbf{v}e^{rt} \quad \text{implies} \quad r\mathbf{v} = A\mathbf{v}.$$

Thus $\mathbf{v}$ is an eigenvector with eigenvalue r.

Note since the equation is homogeneous and linear,

linear combinations of solutions are also solutions:

Suppose $\mathbf{x} = \mathbf{f_1}(t)$ and $\mathbf{x} = \mathbf{f_2}(t)$ are solutions to $\mathbf{x}' = A\mathbf{x}$.

Then $\mathbf{f_1}' = A\mathbf{f_1}$ and $\mathbf{f_2}' = A\mathbf{f_2}$

Thus $[c_1\mathbf{f_1} + c_2\mathbf{f_2}]' = c_1\mathbf{f_1}' + c_2\mathbf{f_2}' = c_1A\mathbf{f_1} + c_2A\mathbf{f_2} = A(c_1\mathbf{f_1} + c_2\mathbf{f_2})$.

Suppose an object moves in the 2D plane (the x_1, x_2 plane) so that it is at the point $(x_1(t), x_2(t))$ at time t. Suppose the object's velocity is given by

$$x_1'(t) = 4x_1 + x_2,$$
$$x_2'(t) = 5x_1$$

Or in matrix form $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 4 & 1 \\ 5 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

To solve, find eigenvalues and corresponding eigenvectors:

$$\begin{vmatrix} 4-r & 1 \\ 5 & -r \end{vmatrix} = (4-r)(-r) - 5 = r^2 - 4r - 5 = (r-5)(r+1).$$

Thus $r = -1, 5$ are eigenvalues.

Eigenvectors associated to eigenvalue $r = -1$: $\begin{pmatrix} 5 & 1 \\ 5 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & \frac{1}{5} \\ 0 & 0 \end{pmatrix}$

Thus x_2 is free and $x_1 + \frac{1}{5}x_2 = 0$

Hence the eigenspace corresponding to $r = -1$ is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{5}x_2 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} -\frac{1}{5} \\ 1 \end{pmatrix}$$

Thus $\begin{pmatrix} -1 \\ 5 \end{pmatrix}$ is an eigenvector with eigenvalue $r = -1$

Hence $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \end{pmatrix} e^{-t}$ is a solution.

E. vectors associated to e. value $r = 5$: $\begin{pmatrix} -1 & 1 \\ 5 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Thus $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector with eigenvalue $r = 5$
since it is a nonzero solution to the above equation.

Hence $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{5t}$ is also a solution.

Hence the general solutions is $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} -1 \\ 5 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{5t}$

Or in non-matrix form: $x_1(t) = -c_1 e^{-t} + c_2 e^{5t}$
$x_2(t) = 5c_1 e^{-t} + c_2 e^{5t}$

IVP: $x_1(t_0) = x_1^0$**,** $x_2(t_0) = x_2^0$

Solve for c_1, c_2: $\quad \begin{pmatrix} x_1^0 \\ x_2^0 \end{pmatrix} = c_1 \begin{pmatrix} -1 \\ 5 \end{pmatrix} e^{-t_0} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{5t_0}$

Or in non-matrix form:

$$\begin{aligned} x_1^0 &= -c_1 e^{-t_0} + c_2 e^{5t_0} \\ x_2^0 &= 5c_1 e^{-t_0} + c_2 e^{5t_0} \end{aligned}$$

Or in matrix form:

$$\begin{pmatrix} x_1^0 \\ x_2^0 \end{pmatrix} = \begin{pmatrix} -e^{-t_0} & e^{5t_0} \\ 5e^{-t_0} & e^{5t_0} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

Thus unique solution iff

$$W[\mathbf{f_1}, \mathbf{f_2}](t_0) = \begin{vmatrix} -e^{-t_0} & e^{5t_0} \\ 5e^{-t_0} & e^{5t_0} \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ 5 & 1 \end{vmatrix} e^{4t_0} = (-1-5)e^{4t_0} \neq 0$$

where $\mathbf{f_1}(t) = \begin{pmatrix} -1 \\ 5 \end{pmatrix} e^{-t}, \quad \mathbf{f_2}(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{5t}$ and

$W[\mathbf{f_1}, \mathbf{f_2}](t_0)$ is the Wronskian of these two vector functions evaluated at t_0.

Note: there is a unique solution to IVP iff the solutions $\mathbf{f_1}, \mathbf{f_1}$ are linearly independent iff the vectors $\begin{pmatrix} -1 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ are linearly independent. But since these vectors have different eigenvalues, we know from linear algebra, that they are linearly independent.

Since we have 3 variables, we can graph a solution to an IVP in $\mathbf{R^3}$. However, sometimes we are interested in how

x_1 varies with t: $\ x_1 = -c_1 e^{-t} + c_2 e^{5t}$

x_2 varies with t: $\ x_2 = 5c_1 e^{-t} + c_2 e^{5t}$

x_2 varies with x_1: Often it is the last pair we are interested in (for example, location of object in above example or predator vs prey or see other examples in 7.1).

$$x_1 = -c_1e^{-t} + c_2e^{5t}$$
$$x_2 = 5c_1e^{-t} + c_2e^{5t}$$

implies $x_2 - x_1 = 6c_1e^{-t}$, $\qquad 5x_1 + x_2 = 6c_2e^{5t} = 6c_2(e^{-t})^{-5}$

Thus $5x_1 + x_2 = 6c_2\left(\frac{x_2 - x_1}{6c_1}\right)^{-5}$ is an implicit solution for x_1, x_2.

To see how x_2 varies with x_1, it is easiest to draw the direction field for the x_1, x_2 plane (the phase plane):

$$\frac{dx_1}{dt} = 4x_1 + x_2,$$
$$\frac{dx_2}{dt} = 5x_1$$

Thus $\frac{\frac{dx_2}{dt}}{\frac{dx_1}{dt}} = \frac{dx_2}{dx_1} = \frac{5x_1}{4x_1 + x_2}$

The graph of a solution to an IVP in the x_1, x_2 plane is called a trajectory.

Some obvious trajectories:

The general solutions is $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1\begin{pmatrix} -1 \\ 5 \end{pmatrix}e^{-t} + c_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{5t}$

IVP: If $\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$, then $c_1 = 1$ and $c_2 = 0$.

Thus $x_1 = -e^{-t}$ and $x_2 = 5e^{-t}$. Thus $x_2 = -5x_1$.

Suppose $x_2 = -5x_1$: $\frac{dx_2}{dx_1} = \frac{5x_1}{4x_1 + x_2} = \frac{5x_1}{4x_1 + -5x_1} = -5$.

Recall $\begin{pmatrix} -1 \\ 5 \end{pmatrix}$ is an eigenvector.

IVP: If $\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, then $c_1 = 0$ and $c_2 = 1$.

Thus $x_1 = e^{5t}$ and $x_2 = e^{5t}$. Thus $x_2 = x_1$.

Suppose $x_2 = 1x_1$: $\frac{dx_2}{dx_1} = \frac{5x_1}{4x_1+x_2} = \frac{5x_1}{4x_1+x_1} = 1$.

Recall $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector.

Suppose $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

Suppose $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = r_1 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = r_2 \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$

Then general solution is $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = k_1 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} e^{r_1 t} + k_2 \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} e^{r_2 t}$

Observe $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} av_1 + bv_2 \\ cv_1 + dv_2 \end{pmatrix} = \begin{pmatrix} r_1 v_1 \\ r_1 v_2 \end{pmatrix}$

IVP: If $\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, then $k_1 = 1$ and $k_2 = 0$.

Thus $x_1 = v_1 e^{r_1 t}$ and $x_2 = v_2 e^{r_1 t}$. Thus $x_2 = \frac{v_2}{v_1} x_1$.

Similarly, if $k_1 = 0$, $x_2 = \frac{w_2}{w_1} x_1$.

Section 3.3: If $b^2 - 4ac < 0$, :

Changed format of $y = c_1e^{r_1t} + c_2e^{r_2t}$ to linear combination of real-valued functions instead of complex valued functions by using Euler's formula:

$$e^{it} = cos(t) + isin(t)$$

Hence $e^{(d+in)t} = e^{dt}e^{int} = e^{dt}[cos(nt) + isin(nt)]$

Let $r_1 = d + in$, $r_2 = d - in$

$y = c_1e^{r_1t} + c_2e^{r_2t} = c_1e^{(d+in)t} + c_2e^{(d-in)t} = c_1e^{dt}e^{int} + c_2e^{dt}e^{-int}$

$= c_1e^{dt}[cos(nt) + isin(nt)] + c_2e^{dt}[cos(-nt) + isin(-nt)]$

$= c_1e^{dt}cos(nt) + ic_1e^{dt}sin(nt) + c_2e^{dt}cos(nt) - ic_2e^{dt}sin(nt)$

$=(c_1 + c_2)e^{dt}cos(nt) + i(c_1 - c_2)e^{dt}sin(nt)$

$$= k_1e^{dt}cos(nt) + k_2e^{dt}sin(nt) = e^{dt}[k_1cos(nt) + k_2sin(nt)]$$

Section 7.6: $(a + d)^2 - 4(ad - bc) < 0$. I.e., $r = \lambda \pm i\mu$

Suppose the eigenvector corresponding to this eigenvalue is

$$\begin{bmatrix} v_1 \pm iw_1 \\ v_2 \pm iw_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \pm i \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

Hence the general solutions in unsimplified form:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 \begin{bmatrix} v_1 + iw_1 \\ v_2 + iw_2 \end{bmatrix} e^{(\lambda+i\mu)t} + c_2 \begin{bmatrix} v_1 - iw_1 \\ v_2 - iw_2 \end{bmatrix} e^{(\lambda-i\mu)t}$$

$$= c_1 \begin{bmatrix} v_1 + iw_1 \\ v_2 + iw_2 \end{bmatrix} e^{\lambda t}e^{i\mu t} + c_2 \begin{bmatrix} v_1 - iw_1 \\ v_2 - iw_2 \end{bmatrix} e^{\lambda t}e^{-i\mu t}$$

$$= c_1 \begin{bmatrix} v_1 + iw_1 \\ v_2 + iw_2 \end{bmatrix} e^{\lambda t}[cos(\mu t) + isin(\mu t)] + c_2 \begin{bmatrix} v_1 - iw_1 \\ v_2 - iw_2 \end{bmatrix} e^{\lambda t}[cos(-\mu t) + isin(-\mu t)]$$

$$= c_1 \begin{bmatrix} v_1 + iw_1 \\ v_2 + iw_2 \end{bmatrix} e^{\lambda t}[cos(\mu t) + isin(\mu t)] + c_2 \begin{bmatrix} v_1 - iw_1 \\ v_2 - iw_2 \end{bmatrix} e^{\lambda t}[cos(\mu t) - isin(\mu t)]$$

$$= c_1 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} e^{\lambda t}[cos(\mu t) + isin(\mu t)] + c_1 \begin{bmatrix} iw_1 \\ iw_2 \end{bmatrix} e^{\lambda t}[cos(\mu t) + isin(\mu t)]$$
$$+c_2 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} e^{\lambda t}[cos(\mu t) - isin(\mu t)] - c_2 \begin{bmatrix} iw_1 \\ iw_2 \end{bmatrix} e^{\lambda t}[cos(\mu t) - isin(\mu t)]$$

$$= c_1 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} e^{\lambda t}[cos(\mu t) + isin(\mu t)] + c_1 \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} e^{\lambda t}[icos(\mu t) + i^2 sin(\mu t)]$$
$$+c_2 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} e^{\lambda t}[cos(\mu t) - isin(\mu t)] - c_2 \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} e^{\lambda t}[icos(\mu t) - i^2 sin(\mu t)]$$

$$= (c_1 + c_2) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} e^{\lambda t} cos(\mu t) + i(c_1 - c_2) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} e^{\lambda t} sin(\mu t)$$
$$+i(c_1 - c_2) \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} e^{\lambda t} cos(\mu t) - (c_1 + c_2) \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} e^{\lambda t} sin(\mu t)$$

$$= (c_1 + c_2) \left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} cos(\mu t) - \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} sin(\mu t) \right) e^{\lambda t}$$
$$+i(c_1 - c_2) \left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} sin(\mu t) + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} cos(\mu t) \right) e^{\lambda t}$$

Then general solution is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 \begin{bmatrix} v_1 cos(\mu t) - w_1 sin(\mu t) \\ v_2 cos(\mu t) - w_2 sin(\mu t) \end{bmatrix} e^{\lambda t} + c_2 \begin{bmatrix} v_1 sin(\mu t) + w_1 cos(\mu t) \\ v_2 sin(\mu t) + w_2 cos(\mu t) \end{bmatrix} e^{\lambda t} \blacksquare$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = e^{\lambda t}\left(c_1 \begin{bmatrix} v_1 cos(\mu t) - w_1 sin(\mu t) \\ v_2 cos(\mu t) - w_2 sin(\mu t) \end{bmatrix} + c_2 \begin{bmatrix} v_1 sin(\mu t) + w_1 cos(\mu t) \\ v_2 sin(\mu t) + w_2 cos(\mu t) \end{bmatrix}\right)$$

7.6 Special case: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$A - \lambda I = \begin{vmatrix} a-\lambda & b \\ -b & a-\lambda \end{vmatrix} = (a-\lambda)^2 + b^2 = \lambda^2 - 2a\lambda + a^2 + b^2$$

Thus $\lambda = \frac{2a \pm \sqrt{4a^2 - 4(a^2+b^2)}}{2} = \frac{2a \pm \sqrt{-4b^2}}{2} = a \pm bi$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ implies $\begin{array}{c} x_1' = ax_1 + bx_2 \\ x_2' = -bx_1 + ax_2 \end{array}$

Change to polar coordinates: $r^2 = x_1^2 + x_2^2$ and $tan\theta = \frac{x_2}{x_1}$

Take derivative with respect to t of both equations:

$2rr' = 2x_1x_1' + 2x_2x_2'$ implies

$rr' = x_1(ax_1 + bx_2) + x_2(-bx_1 + ax_2)$

$\quad = ax_1^2 + bx_1x_2 - bx_1x_2 + ax_2^2 = a(x_1^2 + x_2^2) = ar^2$

Thus $rr' = ar^2$ implies $\frac{dr}{dt} = ar$ and thus $r = Ce^{at}$.

$(sec^2\theta)\theta' = \frac{x_1x_2' - x_1'x_2}{x_1^2} = \frac{x_1(-bx_1+ax_2)-(ax_1+bx_2)x_2}{x_1^2}$

$= \frac{-bx_1^2+ax_1x_2-ax_1x_2-bx_2^2}{x_1^2} = \frac{-b(x_1^2+x_2^2)}{x_1^2} = \frac{-b(r^2)}{x_1^2} = -bsec^2\theta$

$(sec^2\theta)\theta' = -bsec^2\theta$ implies $\theta' = -b$ and thus $\theta = -bt + \theta_0$

Change of basis: Let $\mathbf{x} = P\mathbf{y}$. If $\mathbf{x}' = A\mathbf{x}$, then

$[P\mathbf{y}]' = AP\mathbf{y}$ implies $P\mathbf{y}' = AP\mathbf{y}$. Thus $\mathbf{y}' = P^{-1}AP\mathbf{y}$.

Ch 7 and 9

Suppose an object moves in the 2D plane (the x_1, x_2 plane) so that it is at the point $(x_1(t), x_2(t))$ at time t. Suppose the object's velocity is given by

$$x_1'(t) = ax_1 + bx_2,$$
$$x_2'(t) = cx_1 + dx_2$$

Or in matrix form $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

To solve, find eigenvalues and corresponding eigenvectors:

$$\begin{vmatrix} a-r & b \\ c & d-r \end{vmatrix} = (a-r)(d-r) - bc = r^2 - (a+d)r + ad - bc = 0.$$

$$\text{Thus } r = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$$

Case 1: $(a+d)^2 - 4(ad-bc) > 0$

Hence the general solutions is $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} e^{r_1 t} + c_2 \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} e^{r_2 t}$

Case 1a: $r_1 > r_2 > 0$

Case 1b: $r_1 < r_2 < 0$

Case 1c: $r_2 < 0 < r_1$

Case 2: $(a+d)^2 - 4(ad-bc) = 0$

Case 2i: Two independent eigenvectors:

The general solution is $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} e^{rt} + c_2 \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} e^{rt}$

Case 2ii: One independent eigenvectors:

The general solution is $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} e^{rt} + c_2 \left[\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} t + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \right] e^{rt}$

Case 2a: $r > 0$

Case 2b: $r < 0$

Case 3: $(a+d)^2 - 4(ad-bc) < 0$. I.e., $r = \lambda \pm i\mu$

Suppose eigenvector corresponding to eigenvalue is

$$\begin{pmatrix} v_1 \pm iw_1 \\ v_2 \pm iw_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \pm i \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

Then general solution is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} v_1 cos(\mu t) - w_1 sin(\mu t) \\ v_2 cos(\mu t) - w_2 sin(\mu t) \end{pmatrix} e^{\lambda t} + c_2 \begin{pmatrix} v_1 sin(\mu t) + w_1 cos(\mu t) \\ v_2 sin(\mu t) + w_2 cos(\mu t) \end{pmatrix} e^{\lambda t}$$

Case 3a: $\lambda > 0$

Case 3a: $\lambda < 0$

Case 3a: $\lambda = 0$