Mechanical Vibrations:

$$mu''(t) + \gamma u'(t) + ku(t) = F_{external}, \quad m, \gamma, k \geq 0$$

$$\text{IVP: } u(t_0) = u_0, \quad , u'(t_0) = u_1$$

NOTE: Positive direction points DOWN.

m= mass,

k = spring force proportionality constant,

γ = damping force proportionality constant

g = 9.8 m/sec^2 or 32 ft/sec^2.

Weight $= mg \qquad mg - kL = 0, \qquad F_{damping}(t) = -\gamma u'(t)$

A mass of 3kg stretches a spring 4.9 m. If the mass is acted upon by an external force of $40e^{\frac{-t}{3}}$ N in a medium that imparts a viscous force of 10 N when the speed of the mass is 5 m/sec. If the mass is pulled down 1 m and set in motion with an upward velocity of 8 m/sec, describe the motion of the mass.

$m = 3$

$|F_{damping}(t)| = |\gamma u'(t)| \Rightarrow 10 = \gamma(5)$. Thus $\gamma = 2$

$mg - kL = 0$ implies $= \frac{mg}{L} = \frac{3(9.8)}{4.9} = 6$

IVP: $\quad 3u''+2u'+6u = 40e^{\frac{-t}{3}}, \quad u(0) = 1, \quad u'(0) = -8$

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Step 1: Solve homogeneous eqn: $3u'' + 2u' + 6u = 0$

$u = e^{rt}$ implies $3r^2 + 2r + 6 = 0$ implies

$r = \frac{-2 \pm \sqrt{4-4(3)(6)}}{2(3)} = \frac{-2}{2(3)} \pm \frac{\sqrt{4}\sqrt{1-(3)(6)}}{2(3)} = -\frac{1}{3} \pm i\frac{\sqrt{17}}{3}$

Thus general homogeneous solution is

$$u(t) = c_1 e^{-\frac{t}{3}} cos(\frac{\sqrt{17}}{3}t) + c_2 e^{-\frac{t}{3}} sin(\frac{\sqrt{17}}{3}t)$$

Step 2: Find a non-homogeneous soln

Section 3.6: First find Wronskian,

$$W(e^{-\frac{t}{3}} cos(\frac{\sqrt{17}}{3}t), e^{-\frac{t}{3}} sin(\frac{\sqrt{17}}{3}t)).$$

Too much work. Thus,

Section 3.5: Guess $u(t) = Ae^{\frac{-t}{3}}$.

Then $u'(t) = -\frac{A}{3}e^{\frac{-t}{3}}$ and $u''(t) = \frac{A}{9}e^{\frac{-t}{3}}$

$3(\frac{A}{9}e^{\frac{-t}{3}}) + 2(-\frac{A}{3}e^{\frac{-t}{3}}) + 6Ae^{\frac{-t}{3}} = 40e^{\frac{-t}{3}}$

$\frac{A}{3} - \frac{2A}{3} + 6A = 40$ implies $A - 2A + 18A = 17A = 120$.

Thus $A = \frac{120}{17}$ and

hence $u(t) = \frac{120}{17}e^{\frac{-t}{3}}$ is a non-homogeneous soln.

Thus general NON-homogeneous solution is

$$u(t) = e^{-\frac{t}{3}}[c_1 cos(\frac{\sqrt{17}}{3}t) + c_2 sin(\frac{\sqrt{17}}{3}t)] + \frac{120}{17}e^{\frac{-t}{3}}$$

$$u(t) = e^{-\frac{t}{3}}[c_1 cos(\frac{\sqrt{17}}{3}t) + c_2 sin(\frac{\sqrt{17}}{3}t) + \frac{120}{17}]$$

Step 3: Use initial values to find c_1 and c_2.

$$u(t) = e^{-\frac{t}{3}}[c_1 cos(\frac{\sqrt{17}}{3}t) + c_2 sin(\frac{\sqrt{17}}{3}t) + \frac{120}{17}]$$

$$u'(t) = -\frac{1}{3}e^{-\frac{t}{3}}[c_1 cos(\frac{\sqrt{17}}{3}t) + c_2 sin(\frac{\sqrt{17}}{3}t) + \frac{120}{17}]$$

$$+e^{-\frac{t}{3}}[-c_1\frac{\sqrt{17}}{3}sin(\frac{\sqrt{17}}{3}t) + c_2\frac{\sqrt{17}}{3}cos(\frac{\sqrt{17}}{3}t)]$$

$u(0) = 1$: $\quad 1 = c_1(1) + c_2(0) + \frac{120}{17}$

implies $c_1 = 1 - \frac{120}{17} = -\frac{103}{17}$.

$u'(0) = -8$: $\quad -8 = -\frac{1}{3}[c_1 + \frac{120}{17}] + c_2\frac{\sqrt{17}}{3}$

$$-24 = -[-\frac{103}{17} + \frac{120}{17}] + c_2\sqrt{17}$$

$-24 = -[\frac{17}{17}] + c_2\sqrt{17}$ implies $c_2\sqrt{17} = -23$ and thus $c_2 = -\frac{23}{\sqrt{17}} = -\frac{23\sqrt{17}}{17}$

Thus solution to IVP is

$$u(t) = e^{-\frac{t}{3}}[-\frac{103}{17}cos(\frac{\sqrt{17}}{3}t) - \frac{23\sqrt{17}}{17}sin(\frac{\sqrt{17}}{3}t) + \frac{120}{17}]$$

Thus solution to IVP is

$$u(t) = e^{-\frac{t}{3}}[-\frac{103}{17}cos(\frac{\sqrt{17}}{3}t) - \frac{23\sqrt{17}}{17}sin(\frac{\sqrt{17}}{3}t) + \frac{120}{17}]$$

Simplify:

$$u(t) = e^{-\frac{t}{3}}[Rcos\delta cos(\frac{\sqrt{17}}{3}t) + Rsin\delta sin(\frac{\sqrt{17}}{3}t) + \frac{120}{17}]$$

$$u(t) = e^{-\frac{t}{3}}[Rcos(\frac{\sqrt{17}}{3}t - \delta) + \frac{120}{17}]$$

where $Rcos\delta = c_1 = -\frac{103}{17}$ and $Rsin\delta = c_2 = -\frac{23\sqrt{17}}{17}$.

Thus $R = \sqrt{R^2cos^2\delta + R^2sin^2\delta} = \sqrt{c_1^2 + c_2^2}$

$$R = \sqrt{(-\frac{103}{17})^2 + (-\frac{23\sqrt{17}}{17})^2} = \frac{\sqrt{19602}}{17} = \frac{\sqrt{2(3)^4(11)^2}}{17} = \frac{99\sqrt{2}}{17}$$

and $\frac{c_2}{c_1} = \frac{Rsin\delta}{Rcos\delta} = tan\delta$. Thus $\delta = tan^{-1}(\frac{c_2}{c_1})$, sort of – you must choose the correct quadrant based on the signs of c_1 and c_2.

$\delta = tan^{-1}(\frac{23\sqrt{17}}{103}) + \pi \sim 222.64^o \sim 3.8858$ radians

$\sim 1.24\pi$ radians $\sim \frac{5\pi}{4}$ radians.

Simplified answer to IVP:

$$u(t) = e^{-\frac{t}{3}}[\frac{99\sqrt{2}}{17}cos(\frac{\sqrt{17}}{3}t - (tan^{-1}(\frac{23\sqrt{17}}{103}) + \pi)) + \frac{120}{17}]$$

Approximation of solution:

$$u(t) \sim e^{-\frac{t}{3}}[\frac{99\sqrt{2}}{17}cos(\frac{\sqrt{17}}{3}t - \frac{5\pi}{4}) + \frac{120}{17}]$$