3.7/8 Mechanical Vibrations:

$$mu''(t) + \gamma u'(t) + ku(t) = F_{external}, \quad m, \gamma, k \geq 0$$

$$mg - kL = 0, \qquad F_{damping}(t) = -\gamma u'(t)$$

m= mass,

k = spring force proportionality constant,

γ = damping force proportionality constant

g = 9.8 m/sec^2 or 32 ft/sec^2. Weight = mg.

Electrical Vibrations:

Voltage drop across inductor + resistor + capacitor
= the supplied voltage

$$L\frac{dI(t)}{dt} + RI(t) + \frac{1}{C}Q(t) = E(t), \quad L, R, C \geq 0 \text{ and } I = \frac{dQ}{dt}$$

$$LQ''(t) + RQ'(t) + \frac{1}{C}Q(t) = E(t)$$

L = inductance (henrys),

R = resistance (ohms)

C = capacitance (farads)

$Q(t)$ = charge at time t (coulombs)

$I(t)$ = current at time t (amperes)

$E(t)$ = impressed voltage (volts).

1 volt = 1 ohm · 1 ampere = 1 coulomb / 1 farad =
1 henry · 1 amperes/ 1 second

Trig background:

$$cos(y \mp x) = cos(x \mp y) = cos(x)cos(y) \pm sin(x)sin(y)$$

Let $c_1 = Rcos(\delta)$, $c_2 = Rsin(\delta)$ in

$$\begin{aligned} &c_1 cos(\omega_0 t) + c_2 sin(\omega_0 t) \\ &\qquad = Rcos(\delta)cos(\omega_0 t) + Rsin(\delta)sin(\omega_0 t) \\ &\qquad = Rcos(\omega_0 t - \delta) \end{aligned}$$

Amplitude $= R$

frequency $= \omega_0$ (measured in radians per unit time).

period $= \frac{2\pi}{\omega_0}$ phase (displacement) $= \delta$

$c_1 = Rcos(\delta)$, $c_2 = Rsin(\delta)$ implies

$$\begin{aligned} c_1^2 + c_2^2 &= R^2 cos^2(\delta) + R^2 sin^2(\delta) \\ &= R^2(cos^2(\delta) + sin^2(\delta)) = R^2 \end{aligned}$$

and $\frac{Rsin(\delta)}{Rcos(\delta)} = tan(\delta) = \frac{c_2}{c_1}$

BUT easier to plot to convert Euclidean coordinates $(c_1, c_2) = (Rcos(\delta), Rsin(\delta))$ into polar coordinates $(R, \delta) =$ (length, angle).

3.7: Homogeneous equation (no external force):

$$mu''(t) + \gamma u'(t) + ku(t) = 0, \quad m, \gamma, k \geq 0$$

$r_1, r_2 = \frac{-\gamma \pm \sqrt{\gamma^2 - 4km}}{2m}$

Critical damping: $\gamma = 2\sqrt{km}$

$\gamma^2 - 4km = 0$: $u(t) = (c_1 + c_2 t)e^{r_1 t}$

Note $r_1 = -\frac{\gamma}{2m} < 0$. **Thus** $u(t) \to 0$ **as** $t \to \infty$

Overdamped: $\gamma > 2\sqrt{km}$

$\gamma^2 - 4km > 0$: $u(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

Note $r_1, r_2 < 0$. **Thus** $u(t) \to 0$ **as** $t \to \infty$

Example $u(t) = 4e^{-t} - 3e^{-2t}$

If $t > 0$, $4e^{-t} > 3e^{-2t}$

As $t \to \infty$, $e^{-2t} \to 0$ faster than $e^{-t} \to 0$

If $t < 0$, $4e^{-t} < 3e^{-2t}$

As $t \to -\infty$, $e^{-2t} \to \infty$ faster than $e^{-t} \to \infty$

Underdamped: $\gamma < 2\sqrt{km}$

$$\gamma^2 - 4km < 0\text{: } u(t) = e^{-\frac{\gamma t}{2m}}(c_1 cos\mu t + c_2 sin\mu t)$$
$$= e^{-\frac{\gamma t}{2m}} Rcos(\mu t - \delta)$$
$$\text{where } c_1 = Rcos(\delta),\ c_2 = Rsin(\delta)$$

μ = quasi frequency, $\frac{2\pi}{\mu}$ = quasi period

Note if $\gamma \neq 0$, **then** $u(t) \to 0$ **as** $t \to \infty$

Note if $\gamma = 0$, then

NOTE if $\gamma \neq 0$, then homogeneous solution goes to 0 as $t \to \infty$.

Thus initial values have very little effect on the long-term behaviour of solution if $\gamma \neq 0$.

Note: The larger γ is, the faster the homogeneous solution goes to 0 as $t \to \infty$.

3.8: $F_{external} \neq 0$

$$mu''(t) + \gamma u'(t) + ku(t) = F_{external}, \quad m, \gamma, k \geq 0$$

General solution: $u(t) = c_1\phi_1 + c_2\phi_2 + \psi$

where ϕ_1, ϕ_2 are homogeneous solutions and ψ is a non-homogeneous solution.

NOTE if $\gamma \neq 0$, then homogeneous solution
$c_1\phi_1 + c_2\phi_2$ goes to 0 as $t \to \infty$.

Thus if $\gamma \neq 0$, then $u(t) \to \psi$ as $t \to \infty$.

No damping ($\gamma = 0$) example: $u'' + u = cos(t)$

Step 1: Solve homogeneous $u'' + u = 0$

$r^2 + 1 = 0$ implies $r = \pm i$

Homogeneous solution $u(t) = c_1 cos(t) + c_2 sin(t)$

Step 2: Find a non-homogeneous solution.

Guess $u(t) =$

Plug in plus lots of work implies $A = 0$ and $B = \frac{1}{2}$

Thus general non-homogeneous solution:

$$u(t) = c_1 cos(t) + c_2 sin(t) + \frac{1}{2} t sin(t)$$

No damping example $u''+u = cos(\omega t)$ where $\omega \neq 1$.

Step 1: Solve homogeneous $u'' + u = 0$

$$r^2 + 1 = 0 \text{ implies } r = \pm i$$

$$\text{Homogeneous solution } u(t) = c_1 cos(t) + c_2 sin(t)$$

Step 2: Find a non-homogeneous solution.

Since $\omega \neq 1$, guess $u(t) = Acos(\omega t)$

$$u'(t) = -A\omega sin(\omega t)$$

$$u''(t) = -A\omega^2 cos(\omega t)$$

Plug into $u'' + u = cos(\omega t)$:

$$-A\omega^2 cos(\omega t) + Acos(\omega t) = cos(\omega t)$$

$$-A\omega^2 + A = 1. \text{ Thus } A(1 - \omega^2) = 1$$

Hence $A = \frac{1}{1-\omega^2}$

Thus general solution is

$$u(t) = c_1 cos(t) + c_2 sin(t) + \frac{1}{1-\omega^2} cos(\omega t)$$

NOTE: Since we do not have damping, we do NOT have a transient solution.

BUT if ω is close to 1, then $\frac{1}{1-\omega^2}$ is large and the term $\frac{1}{1-\omega^2} cos(\omega t)$ dominates.

Trig background:

$$cos(y \mp x) = cos(x \mp y) = cos(x)cos(y) \pm sin(x)sin(y)$$

$$cos(u) + cos(v) = 2cos(\frac{u+v}{2})cos(\frac{u-v}{2})$$

$$cos(u) - cos(v) = -2sin(\frac{u+v}{2})sin(\frac{u-v}{2})$$

$$sin(u) + sin(v) = 2sin(\frac{u+v}{2})cos(\frac{u-v}{2})$$

$$sin(u) - sin(v) = sin(u) + sin(-v) = 2sin(\frac{u-v}{2})cos(\frac{u+v}{2})$$

Derivation:

Let $x = (\frac{u+v}{2})$ and $y = (\frac{u-v}{2})$

$$cos(u) = cos((\frac{u+v}{2}) + (\frac{u-v}{2}))$$

$$= cos(\frac{u+v}{2})cos(\frac{u-v}{2}) - sin(\frac{u+v}{2})sin(\frac{u-v}{2})$$

$$cos(v) = cos((\frac{u+v}{2}) - (\frac{u-v}{2}))$$

$$= cos(\frac{u+v}{2})cos(\frac{u-v}{2}) + sin(\frac{u+v}{2})sin(\frac{u-v}{2})$$

Ex: $u(t) = cos(t) + cos(3t) =$

Graph:

No damping example $mu'' + ku = cos(\omega t)$.

Step 1: Solve homogeneous $mu'' + ku = 0$

$$mr^2 + k = 0 \text{ implies } r = \pm i\sqrt{\frac{k}{m}}$$

Let $\omega_0 = \sqrt{\frac{k}{m}}$. Then $r = \pm i\omega_0$ and

Homogeneous solution $u(t) = c_1 cos(\omega_0 t) + c_2 sin(\omega_0 t)$

Step 2: Find a non-homogeneous solution.

IF $\omega = \omega_0$, guess $u(t) = t[Acos(\omega t) + Bsin(\omega t)]$

Plug in plus lots of work implies $A = 0$ and $B = \frac{1}{2\sqrt{mk}}$

Thus general non-homogeneous solution:

$$u(t) = c_1 cos(\omega_0 t) + c_2 sin(\omega_0 t) + \frac{1}{2\sqrt{mk}} t sin(t)$$

IF $\omega \neq \omega_0$, guess $u(t) = Acos(\omega t)$

$$u'(t) = -A\omega sin(\omega t)$$

$$u''(t) = -A\omega^2 cos(\omega t)$$

Plug into $mu'' + ku = cos(\omega t)$:

$$-mA\omega^2 cos(\omega t) + kAcos(\omega t) = cos(\omega t)$$

$$-mA\omega^2 + kA = 1. \text{ Thus } A(k - m\omega^2) = 1$$

Hence $A = \frac{1}{k-m\omega^2}$

Thus general solution is

$$u(t) = c_1 cos(\omega_0 t) + c_2 sin(\omega_0 t) + \frac{1}{k-m\omega^2} cos(\omega t)$$

NOTE: Since we do not have damping, we do NOT have a transient solution.

BUT if ω^2 is close to $\frac{k}{m}$, then $\frac{1}{k-m\omega^2}$ is large and the term $\frac{1}{k-m\omega^2} cos(\omega t)$ dominates.

Example **with small damping** $(\gamma = \frac{1}{8} < 2\sqrt{km})$**:**

Compare book examples (see slides)

$$u'' + \frac{1}{8}u' + u = 3cos(0.3t), \quad u(0) = 2, \quad u'(0) = 0$$

$$u'' + \frac{1}{8}u' + u = 3cos(t), \quad u(0) = 2, \quad u'(0) = 0$$

$$u'' + \frac{1}{8}u' + u = 3cos(2t), \quad u(0) = 2, \quad u'(0) = 0$$

Approximate midterm grades

A $\geq$ 52

A- 50-51

B+ 48-49

B 42-47

B- 40-41

C+ 38-39

C 28-37

C- 22-27

D 20-21

F 0-19