3.6 Variation of Parameters   Solve $y'' - 2y' + y = e^t ln(t)$

1) Find homogeneous solutions: Solve $y'' - 2y' + y = 0$

Guess: $y = e^{rt}$, then $y' = re^{rt}$, $y'' = r^2 e^{rt}$, and

$$r^2 e^{rt} - 2re^{rt} + e^{rt} = 0 \text{ implies } r^2 - 2r + 1 = 0$$

$(r-1)^2 = 0$, and hence $r = 1$

General homogeneous solution: $y = c_1 e^t + c_2 te^t$

since have two linearly independent solutions: $\{e^t, te^t\}$

2.) Find a non-homogeneous solution:

Sect. 3.5 method: Educated guess

Sect. 3.6: **Guess** $y = u_1(t)e^t + u_2(t)te^t$ **and solve for** u_1 **and** u_2

$$u_1(t) = \int \frac{\begin{vmatrix} 0 & \phi_2 \\ 1 & \phi_2' \end{vmatrix}}{\begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix}} g(t)dt = -\int \frac{\phi_2(t)g(t)}{W(\phi_1,\phi_2)}dt = -\int \frac{(te^t)(e^t ln(t))}{e^{2t}}dt$$

$$= -\int tln(t) = -[\frac{t^2 ln(t)}{2} - \int \frac{t}{2}] = -\frac{t^2 ln(t)}{2} + \frac{t^2}{4}$$

$$u_2(t) = \int \frac{\begin{vmatrix} \phi_1 & 0 \\ \phi_1' & 1 \end{vmatrix}}{\begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix}} g(t)dt = \int \frac{\phi_1(t)g(t)}{W(\phi_1,\phi_2)}dt = \int \frac{(e^t)(e^t ln(t))}{e^{2t}}dt$$

$$= \int ln(t) = tln(t) - t$$

$$W(\phi_1, \phi_2) = \begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix} = \begin{vmatrix} e^t & te^t \\ e^t & e^t + te^t \end{vmatrix}$$

$u = ln(t) \quad dv = tdt$
$du = \frac{dt}{t} \quad v = \frac{t^2}{2}$

$u = ln(t) \quad dv = dt$
$du = \frac{dt}{t} \quad v = t$

General solution : $y = c_1 e^t + c_2 te^t + (-\frac{t^2 ln(t)}{2} + \frac{t^2}{4})e^t + (tln(t) - t)te^t$

which simplifies to $y = c_1 e^t + c_2 te^t + (\frac{ln(t)}{2} - \frac{3}{4})t^2 e^t$

Solve $y''+p(t)y'+q(t)y = g(t)$ where $y = c_1\phi_1(t)+c_2\phi_2(t)$ is solution to homogeneous equation $y'' + p(t)y' + q(t)y = 0$

Guess $y = u_1(t)\phi_1(t) + u_2(t)\phi_2(t)$

$y = u_1\phi_1 + u_2\phi_2$ implies $y' = u_1\phi_1' + u_1'\phi_1 + u_2\phi_2' + u_2'\phi_2$

Two unknown functions, u_1 and u_2, but only one equation ($y'' + p(t)y' + q(t)y = g(t)$). Thus might be OK to choose 2nd eq'n.

Avoid 2nd derivative in y'': **Choose** $u_1'\phi_1 + u_2'\phi_2 = 0$

$y' = u_1\phi_1' + u_2\phi_2'$ implies $y'' = u_1\phi_1'' + u_1'\phi_1' + u_2\phi_2'' + u_2'\phi_2'$

Plug into $y'' + p(t)y' + q(t)y = g(t)$:

$$u_1\phi_1'' + u_1'\phi_1' + u_2\phi_2'' + u_2'\phi_2' + p(u_1\phi_1' + u_2\phi_2') + q(u_1\phi_1 + u_2\phi_2) = g$$

$$u_1\phi_1'' + u_1'\phi_1' + u_2\phi_2'' + u_2'\phi_2' + pu_1\phi_1' + pu_2\phi_2' + qu_1\phi_1 + qu_2\phi_2 = g$$

$$u_1\phi_1'' + pu_1\phi_1' + qu_1\phi_1 + u_1'\phi_1' + u_2\phi_2'' + pu_2\phi_2' + qu_2\phi_2 + u_2'\phi_2' = g$$

$$u_1(\phi_1'' + p\phi_1' + q\phi_1) + u_1'\phi_1' + u_2(\phi_2'' + p\phi_2' + q\phi_2) + u_2'\phi_2' = g$$

ϕ_1, ϕ_2 are homogeneous solutions. Thus $\phi_i'' + p\phi_i' + q\phi_i = 0$.

Hence $u_1(0) + u_1'\phi_1' + u_2(0) + u_2'\phi_2' = g$

Thus we have 2 eqns to find 2 unknowns, the functions u_1 and u_2:

$$\begin{array}{r} u_1'\phi_1 + u_2'\phi_2 = 0 \\ u_1'\phi_1' + u_2'\phi_2' = g \end{array} \text{ implies } \begin{bmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ g \end{bmatrix}$$

Cramer's rule: $u_1'(t) = \dfrac{\begin{vmatrix} 0 & \phi_2 \\ g & \phi_2' \end{vmatrix}}{\begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix}}$ and $u_2'(t) = \dfrac{\begin{vmatrix} \phi_1 & 0 \\ \phi_1' & g \end{vmatrix}}{\begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix}}$

Sect.3.6: **Guess** $y = u_1(t)e^t + u_2(t)te^t$ **and solve for** u_1 **and** u_2

$y' = u_1'e^t + u_1e^t + u_2'te^t + u_2(e^t + te^t) = e^{2t} + te^{2t} - te^{2t} - e^{2t}.$

Two unknown functions, u_1 and u_2, but only one equation ($y'' - 2y' + y = e^t ln(t)$). Thus might be OK to choose 2nd eq'n.

Avoid 2nd derivative in y''**:** **Choose** $u_1'e^t + u_2'te^t = 0$

Hence $y' = u_1e^t + u_2(e^t + te^t)$.

and $y'' = u_1'e^t + u_1e^t + u_2'(e^t + te^t) + u_2(e^t + e^t + te^t)$.
$= u_1'e^t + u_1e^t + u_2'e^t + u_2'te^t + u_2(2e^t + te^t)$.
$= u_1e^t + u_2'e^t + u_2(2e^t + te^t)$.

Solve $y'' - 2y' + y = e^t ln(t)$

$u_1e^t + u_2'e^t + u_2(2e^t + te^t) - 2[u_1e^t + u_2(e^t + te^t)] + u_1e^t + u_2te^t = e^t ln(t)$

$u_2'e^t + 2u_2e^t + u_2te^t - 2u_2e^t - 2u_2te^t + u_2te^t = e^t ln(t)$

$u_2' = ln(t)$ or in other words, $\frac{du_2}{dt} = ln(t)$

Thus $\int du_2 = \int ln(t)dt$

$u_2 = tln(t) - t$. Note only need one solution, so don't need $+C$.

$y = u_1(t)e^t + [tln(t) - t]te^t$

$u_1'e^t + u_2'te^t = 0$. Thus $u_1' + u_2't = 0$. Hence $u_1' = -u_2't = -tln(t)$

Thus $u_1 = -\int tln(t)dt = -\frac{t^2 ln(t)}{2} + \frac{t^2}{4}$

Thus the general solution is

$$y = c_1e^t + c_2te^t + (-\frac{t^2 ln(t)}{2} + \frac{t^2}{4})e^t + (tln(t) - t)te^t$$