Solve $y'' - 4y' - 5y = 4sin(3t), \;\; y(0) = 6, \, y'(0) = 7.$

1.) Find the general solution to $y'' - 4y' - 5y = 0$:

Guess $y = e^{rt}$ for HOMOGENEOUS equation:

$y' = re^{rt}, \, y' = r^2 e^{rt}$

$y'' - 4y' - 5y = 0$

$r^2 e^{rt} - 4re^{rt} - 5e^{rt} = 0$

$e^{rt}(r^2 - 4r - 5) = 0$

$e^{rt} \neq 0$, thus can divide both sides by e^{rt}:

$$r^2 - 4r - 5 = 0$$

$(r+1)(r-5) = 0$. Thus $r = -1, 5$.

Thus $y = e^{-t}$ and $y = e^{5t}$ are both solutions to HOMOGENEOUS equation.

Thus the general solution to the 2nd order linear HOMOGENEOU equation is

$$y = c_1 e^{-t} + c_2 e^{5t}$$

2.) Find a solution to $ay'' + by' + cy = 4sin(3t)$:

Guess $y = Asin(3t) + Bcos(3t)$

$y' = 3Acos(3t) - 3Bsin(3t)$

$y'' = -9Asin(3t) - 9Bcos(3t)$

$y'' - 4y' - 5y = 4sin(3t)$

$$\begin{array}{ccccc} -9Asin(3t) & - & 9Bcos(3t) & & \\ 12Bsin(3t) & - & 12Acos(3t) & & \\ -5Asin(3t) & - & 5cos(3t) & & \\ \hline (12B - 14A)sin(3t) & - & (-14B - 12A)cos(3t) & = & 4sin(3t) \end{array}$$

Since $\{sin(3t), cos(3t)\}$ is a linearly independent set:

$12B - 14A = 4$ and $-14B - 12A = 0$

Thus $A = -\frac{14}{12}B = -\frac{7}{6}B$ and

$12B - 14(-\frac{7}{6}B) = 12B + 7(\frac{7}{3}B) = \frac{36+49}{3}B = \frac{85}{3}B = 4$

Thus $B = 4(\frac{3}{85}) = \frac{12}{85}$ and $A = -\frac{7}{6}B = -\frac{7}{6}(\frac{12}{85}) = -\frac{14}{85}$

Thus $y = (-\frac{14}{85})sin(3t) + \frac{12}{85}cos(3t)$ is one solution to the non-homogeneous equation.

Thus the general solution to the 2nd order linear nonhomogeneous equation is

$$y = c_1e^{-t} + c_2e^{5t} - (\frac{14}{85})sin(3t) + \frac{12}{85}cos(3t)$$

3.) If initial value problem:

Once general solution is known, can solve initial value problem (i.e., use initial conditions to find c_1, c_2).

NOTE: you must know the GENERAL solution to the ODE BEFORE you can solve for the initial values. The homogeneous solution and the one nonhomogeneous solution found in steps 1 and 2 above do NOT need to satisfy the initial values.

Solve $y'' - 4y' - 5y = 4sin(3t)$, $y(0) = 6$, $y'(0) = 7$.

General solution: $y = c_1e^{-t} + c_2e^{5t} - (\frac{14}{85})sin(3t) + \frac{12}{85}cos(3t)$

Thus $y' = -c_1e^{-t} + 5c_2e^{5t} - (\frac{42}{85})cos(3t) - \frac{36}{85}sin(3t)$

$y(0) = 6$: $\quad 6 = c_1 + c_2 + \frac{12}{85} \quad \frac{498}{85} = c_1 + c_2$

$y'(0) = 7$: $\quad 7 = -c_1 + 5c_2 - \frac{42}{85} \quad \frac{637}{85} = -c_1 + 5c_2$

$6c_2 = \frac{498+637}{85} = \frac{1135}{85} = \frac{227}{17}$. Thus $c_2 = \frac{227}{102}$.

$c_1 = \frac{498}{85} - c_2 = \frac{498}{85} - \frac{227}{102} = \frac{2988-1135}{510} = \frac{1853}{510} = \frac{109}{30}$

Thus $y = (\frac{109}{30})e^{-t} + (\frac{227}{102})e^{5t} - (\frac{14}{85})sin(3t) + \frac{12}{85}cos(3t)$.

Partial Check: $y(0) = (\frac{109}{30}) + (\frac{227}{102}) + \frac{12}{85} = 6.$

$$y'(0) = -\frac{109}{30} + 5(\frac{227}{102}) - \frac{42}{85} = 7.$$