3.5: Solving 2nd order linear non-homogeneous DE using method of undetermined coefficients.

Example: Solve $y'' + 4y = 12t + 8sin(2t)$.

Step 1: Solve homogeneous system, $y'' + 4y = 0$

$r^2 + 4 = 0 \quad \Rightarrow \quad r^2 = -4 \quad \Rightarrow \quad r = \pm\sqrt{-4} = 0 \pm 2i$

Hence homogeneous soln is $y = c_1 cos(2t) + c_2 sin(2t)$

Step 2a: Find one solution to $y'' + 4y = 12t$

Possible guess: $y = At + B$. Then $y' = A$ and $y'' = 0$.

Plug in: $0 + 4(At + B) = 12t \Rightarrow 4At + 4B = 12t + 0$

Thus $4A = 12$ and $4B = 0 \quad \Rightarrow \quad A = 3$ and $B = 0$

Thus $y = 3t$ is a solution to $y'' + 4y = 12t$.

Simpler guess: since there is no y' term, we didn't need the B term in our guess. We could have guessed $y = At$ instead for this particular problem (and other analogous problems). If you make similar observations when you do your HW, you can save time when you do comparable problems.

Step 2b: Find one solution to $y'' + 4y = 8sin(2t)$

Incorrect guess: $y = Asin(2t)$. Then $y' = 2Acos(2t)$ and $y'' = -4Asin(2t)$.

Note: since no y' term, did not include a $Bcos(2t)$ term in guess.

Plug in: $-4Asin(2t) + 4Asin(2t) = 8sin(2t)$.

$$\text{Thus } 0 = 8sin(2t).$$

Thus equation has no solution for A. Hence guess is wrong.

Note this guess is wrong because $y = sin(2t)$ is a homogeneous solution. This is why we always solve homogeneous equations first. If a function is a solution to a homogeneous equation, then no constant multiple of that function can be a solution to a non-homogeneous solution since it is a homogeneous solution.

If your normal guess is a homogeneous solution:

$$\text{Multiply it by } t$$

until it is no longer a homogeneous solution.

Incorrect guess: $y = Atsin(2t)$.

Then $y' = Asin(2t) + 2Atcos(2t)$ and

$$y'' = 2Acos(2t) + 2Acos(2t) - 4Atsin(2t)$$
$$= 4Acos(2t) - 4Atsin(2t).$$

Plug into $y'' + 4y = 8sin(2t)$:

$$4Acos(2t) - 4Atsin(2t) + 4Atsin(2t) = 8sin(2t)$$

But this equation has no solution for A. Note we need to add a cosine term to our guess so that we can cancel out the cosine term on LHS:

Better guess: $y = t[Asin(2t) + Bcos(2t)]$.

Best guess: $y = Btcos(2t)$

Then $y' = Bcos(2t) - 2Btsin(2t)$

and $y'' = -2Bsin(2t) - 2Bsin(2t) - 4Btcos(2t)$
$= -4Bsin(2t) - 4Btcos(2t)$

Plug into $y'' + 4y = 8sin(2t)$

$$-4Bsin(2t) - 4Btcos(2t) + 4Btcos(2t) = 8sin(2t)$$

$$-4Bsin(2t) = 8sin(2t) \quad \Rightarrow \quad -4B = 8 \quad \Rightarrow \quad B = -2$$

Thus $y = -2tcos(2t)$ is a solution to

$$y'' + 4y = 8sin(2t)$$

Note: Guessing wrong is NOT a big deal. You can use your wrong guess to determine a correct guess (though guessing right the first time will save you time).

Recall you are looking for ONE solution to your NON-homogeneous equation.

- If you find an infinite number of solns, choose one.

- If your guess gives you one solution, use it.

- If your guess leads to no solutions, than make a different (improved) educated guess.

To find general solution to non-homogeneous LINEAR differential equation: combine all solutions

$$y = c_1cos(2t) + c_2sin(2t) + 3t - 2tcos(2t)$$