Thm: Suppose $c_1\phi_1(t) + c_2\phi_2(t)$ is a general solution to
$$ay'' + by' + cy = 0,$$

If ψ is a solution to
$$ay'' + by' + cy = g(t) \ [*],$$

Then $\psi + c_1\phi_1(t) + c_2\phi_2(t)$ is also a solution to [*].

Moreover if γ is also a solution to [*], then there exist constants c_1, c_2 such that
$$\gamma = \psi + c_1\phi_1(t) + c_2\phi_2(t)$$

Or in other words, $\psi + c_1\phi_1(t) + c_2\phi_2(t)$ is a general solution to [*].

Proof:

Define $L(f) = af'' + bf' + cf$.

Recall L is a linear function.

Let $h = c_1\phi_1(t) + c_2\phi_2(t)$. Since h is a solution to the differential equation, $ay'' + by' + cy = 0$,

Since ψ is a solution to $ay'' + by' + cy = g(t)$, We will now show that $\psi + c_1\phi_1(t) + c_2\phi_2(t) = \psi + h$ is also a solution to [*].

Since γ a solution to $ay'' + by' + cy = g(t)$,

We will first show that $\gamma - \psi$ is a solution to the differential equation $ay'' + by' + cy = 0$.

Since $\gamma - \psi$ is a solution to $ay'' + by' + cy = 0$ and

$c_1\phi_1(t) + c_2\phi_2(t)$ is a general solution to

$$ay'' + by' + cy = 0,$$

there exist constants c_1, c_2 such that

$$\gamma - \psi = \underline{\hspace{8cm}}$$

Thus $\gamma = \psi + c_1\phi_1(t) + c_2\phi_2(t)$.

Thm:
Suppose f_1 is a a solution to $ay'' + by' + cy = g_1(t)$ and f_2 is a a solution to $ay'' + by' + cy = g_2(t)$, then $f_1 + f_2$ is a solution to $ay'' + by' + cy = g_1(t) + g_2(t)$

Proof: Let $L(f) = af'' + bf' + cf$.

Since f_1 is a solution to $ay'' + by' + cy = g_1(t)$,

Since f_2 is a solution to $ay'' + by' + cy = g_2(t)$,

We will now show that $f_1 + f_2$ is a solution to $ay'' + by' + cy = g_1(t) + g_2(t)$.

Sidenote: The proofs above work even if a, b, c are functions of t instead of constants.

Examples: Find a suitable form for ψ for the following differential equations:

1.) $y'' - 4y' - 5y = 4e^{2t}$

2.) $y'' - 4y' - 5y = t^2 - 2t + 1$

3.) $y'' - 4y' - 5y = 4sin(3t)$

4.) $y'' - 5y = 4sin(3t)$

5.) $y'' - 4y' = t^2 - 2t + 1$

6.) $y'' - 4y' - 5y = 4(t^2 - 2t - 1)e^{2t}$

7.) $y'' - 4y' - 5y = 4sin(3t)e^{2t}$

8.) $y'' - 4y' - 5y = 4(t^2 - 2t - 1)sin(3t)e^{2t}$

9.) $y'' - 4y' - 5y = 4sin(3t) + 4sin(3t)e^{2t}$

10.) $y'' - 4y' - 5y$
$= 4sin(3t)e^{2t} + 4(t^2 - 2t - 1)e^{2t} + t^2 - 2t - 1$

11.) $y'' - 4y' - 5y = 4sin(3t) + 5cos(3t)$

12.) $y'' - 4y' - 5y = 4e^{-t}$

To solve $ay'' + by' + cy = g_1(t) + g_2(t) + ...g_n(t)$ [**]

1.) Find the general solution to $ay'' + by' + cy = 0$:

$$c_1\phi_1 + c_2\phi_2$$

2.) For each g_i, find a solution to $ay'' + by' + cy = g_i$:

$$\psi_i$$

This includes plugging guessed solution ψ_i into $ay'' + by' + cy = g_i$.

The general solution to [**] is

$$c_1\phi_1 + c_2\phi_2 + \psi_1 + \psi_2 + ...\psi_n$$

3.) If initial value problem:

Once general solution is known, can solve initial value problem (i.e., use initial conditions to find c_1, c_2).

Solve $y'' - 4y' - 5y = 4sin(3t), \;\; y(0) = 6, \; y'(0) = 7.$

1.) **First solve homogeneous equation:**

Find the general solution to $y'' - 4y' - 5y = 0$:

Guess $y = e^{rt}$ for HOMOGENEOUS equation:

$y' = re^{rt}, \; y' = r^2e^{rt}$

$y'' - 4y' - 5y = 0$

$r^2e^{rt} - 4re^{rt} - 5e^{rt} = 0$

$e^{rt}(r^2 - 4r - 5) = 0$

$e^{rt} \neq 0$, thus can divide both sides by e^{rt}:

$$r^2 - 4r - 5 = 0$$

$(r + 1)(r - 5) = 0$. Thus $r = -1, 5$.

Thus $y = e^{-t}$ and $y = e^{5t}$ are both solutions to LINEAR HOMOGENEOUS equation.

Thus the general solution to the 2nd order LINEAR HOMOGENEOUS equation is

$$y = c_1e^{-t} + c_2e^{5t}$$

2.) **Find one solution to non-homogeneous eq'n:**

Find a solution to $ay'' + by' + cy = 4sin(3t)$:

Guess $y = Asin(3t) + Bcos(3t)$

$$y' = 3Acos(3t) - 3Bsin(3t)$$

$$y'' = -9Asin(3t) - 9Bcos(3t)$$

$y'' - 4y' - 5y = 4sin(3t)$

$$\begin{array}{ccccc} -9Asin(3t) & - & 9Bcos(3t) & & \\ 12Bsin(3t) & - & 12Acos(3t) & & \\ -5Asin(3t) & - & 5cos(3t) & & \\ \hline (12B - 14A)sin(3t) & - & (-14B - 12A)cos(3t) & = & 4sin(3t) \end{array}$$

Since $\{sin(3t), cos(3t)\}$ is a linearly independent set:

$12B - 14A = 4$ and $-14B - 12A = 0$

Thus $A = -\frac{14}{12}B = -\frac{7}{6}B$ and

$12B - 14(-\frac{7}{6}B) = 12B + 7(\frac{7}{3}B) = \frac{36+49}{3}B = \frac{85}{3}B = 4$

Thus $B = 4(\frac{3}{85}) = \frac{12}{85}$ and $A = -\frac{7}{6}B = -\frac{7}{6}(\frac{12}{85}) = -\frac{14}{85}$

Thus $y = (-\frac{14}{85})sin(3t) + \frac{12}{85}cos(3t)$ is one solution to the nonhomogeneous equation.

Thus the general solution to the 2nd order linear non-homogeneous equation is

$$y = c_1e^{-t} + c_2e^{5t} - (\frac{14}{85})sin(3t) + \frac{12}{85}cos(3t)$$

3.) **If initial value problem:**

Once general solution is known, can solve initial value problem (i.e., use initial conditions to find c_1, c_2).

NOTE: you must know the GENERAL solution to the ODE BEFORE you can solve for the initial values. The homogeneous solution and the one nonhomogeneous solution found in steps 1 and 2 above do NOT need to separately satisfy the initial values.

Solve $y'' - 4y' - 5y = 4sin(3t)$, $y(0) = 6$, $y'(0) = 7$.

General solution: $y = c_1e^{-t} + c_2e^{5t} - (\frac{14}{85})sin(3t) + \frac{12}{85}cos(3t)$

Thus $y' = -c_1e^{-t} + 5c_2e^{5t} - (\frac{42}{85})cos(3t) - \frac{36}{85}sin(3t)$

$y(0) = 6$: $\quad 6 = c_1 + c_2 + \frac{12}{85} \quad \frac{498}{85} = c_1 + c_2$

$y'(0) = 7$: $\quad 7 = -c_1 + 5c_2 - \frac{42}{85} \quad \frac{637}{85} = -c_1 + 5c_2$

$6c_2 = \frac{498+637}{85} = \frac{1135}{85} = \frac{227}{17}$. Thus $c_2 = \frac{227}{102}$.

$c_1 = \frac{498}{85} - c_2 = \frac{498}{85} - \frac{227}{102} = \frac{2988-1135}{510} = \frac{1853}{510} = \frac{109}{30}$

Thus $y = (\frac{109}{30})e^{-t} + (\frac{227}{102})e^{5t} - (\frac{14}{85})sin(3t) + \frac{12}{85}cos(3t)$.

Partial Check: $y(0) = (\frac{109}{30}) + (\frac{227}{102}) + \frac{12}{85} = 6$.

$$y'(0) = -\frac{109}{30} + 5(\frac{227}{102}) - \frac{42}{85} = 7.$$

$(e^{-t})'' - 4(e^{-t})' - 5(e^{-t}) = 0$ and $(e^{5t})'' - 4(e^{5t})' - 5(e^{5t}) = 0$