$mu'' + \gamma u' + ku = F_0 cos(\omega t)$

Non-homogeneous solution: Guess $\psi(t) = Acos(\omega t) + Bsin(\omega t)$.

$\psi'(t) = -A\omega sin(\omega t) + B\omega cos(\omega t)$ and $\psi''(t) = -A\omega^2 cos(\omega t) - B\omega^2 sin(\omega t)$.

Plug into $mu'' + \gamma u' + ku = F_0 cos(\omega t)$:

$$m[-A\omega^2 cos(\omega t) - B\omega^2 sin(\omega t)] + \gamma[-A\omega sin(\omega t) + B\omega cos(\omega t)] + k[Acos(\omega t) + Bsin(\omega t)] = F_0 cos(\omega t)$$

$$cos(\omega t)[-mA\omega^2 + \ gB\omega + kA] + sin(\omega t)[-mB\omega^2 - \gamma A\omega + kB] = F_0 cos(\omega t)$$

$$cos(\omega t)[(-m\omega^2 + k)A + \ g\omega B] + sin(\omega t)[(-m\omega^2 + k)B - \gamma\omega A] = F_0 cos(\omega t) + 0 sin(\omega t)$$

$(-m\omega^2 + k)A + \gamma\omega B = F_0$

$(-m\omega^2 + k)B - \gamma\omega A = 0$. Thus $A = \frac{(-m\omega^2+k)B}{\gamma\omega}$. Hence $\frac{(-m\omega^2+k)^2 B}{\gamma\omega} + \gamma\omega B = F_0$.

$(-m\omega^2 + k)^2 B + (\gamma\omega)^2 B = F_0\gamma\omega$ $\qquad$ $[(-m\omega^2 + k)^2 + (\gamma\omega)^2]B = F_0\gamma\omega$

$B = \frac{F_0\gamma\omega}{[(-m\omega^2+k)^2+(\gamma\omega)^2]}$. $\qquad$ Thus $A = \frac{(-m\omega^2+k)F_0\gamma\omega}{\gamma\omega[(-m\omega^2+k)^2+(\gamma\omega)^2]} = \frac{(-m\omega^2+k)F_0}{[(-m\omega^2+k)^2+(\gamma\omega)^2]}$

$\psi(t) = Acos(\omega t) + Bsin(\omega t) \ = Rcos(\omega t - \delta)$.

where $A = Rcos(\delta)$, $B = Rsin(\delta)$ in $Acos(\omega t) + Bsin(\omega t)$. Thus,

$Acos(\omega t) + Bsin(\omega t) \ = Rcos(\delta)cos(\omega t) + Rsin(\delta)sin(\omega t) \ = Rcos(\omega t - \delta)$

where $R = \sqrt{A^2 + B^2} = \sqrt{(\frac{(-m\omega^2+k)F_0}{[(-m\omega^2+k)^2+(\gamma\omega)^2]})^2 + (\frac{F_0\gamma\omega}{[(-m\omega^2+k)^2+(\gamma\omega)^2]})^2}$

$= \frac{F_0}{[(-m\omega^2+k)^2+(\gamma\omega)^2]}\sqrt{((-m\omega^2 + k))^2 + (\gamma\omega)^2}$

$= \frac{F_0}{\sqrt{((-m\omega^2+k))^2+(\gamma\omega)^2}} = \frac{F_0}{\sqrt{(m(\frac{k}{m}-\omega^2))^2+(\gamma\omega)^2}}$

So $R = \frac{F_0}{\sqrt{m^2({\omega_o}^2-\omega^2)^2+(\gamma\omega)^2}}$ for ${\omega_o}^2 = \frac{k}{m}$

Find maximux amplitude

$R(\omega) = \frac{F_0}{\sqrt{m^2({\omega_o}^2-\omega^2)^2+(\gamma\omega)^2}} = F_0[m^2({\omega_o}^2-\omega^2)^2+(\gamma\omega)^2]^{\frac{-1}{2}}$ for ${\omega_o}^2 = \frac{k}{m}$

$R'(\omega) = -\frac{1}{2}F_0[m^2({\omega_o}^2-\omega^2)^2+(\gamma\omega)^2]^{\frac{-3}{2}}[2m^2({\omega_o}^2-\omega^2)(-2\omega)+2\gamma^2\omega]$

Critical point occur when $2m^2({\omega_o}^2-\omega^2)(-2\omega)+2\gamma^2\omega = 0$

$2m^2({\omega_o}^2-\omega^2)(-2\omega)+2\gamma^2\omega = [-2m^2({\omega_o}^2-\omega^2)+\gamma^2)]2\omega = 0$

$\omega = 0$ or $[-2m^2({\omega_o}^2-\omega^2)+\gamma^2)] = 0$

$2m^2({\omega_o}^2-\omega^2) = \gamma^2$ implies ${\omega_o}^2-\omega^2 = \frac{\gamma^2}{2m^2}$ implies $\omega^2 = \omega_o^2 - \frac{\gamma^2}{2m^2}$

Critical pts are $\omega = 0, \pm\sqrt{\omega_o^2-\frac{\gamma^2}{2m^2}} = \pm\sqrt{\omega_o^2-\frac{\gamma^2 k}{2km^2}} = \pm\omega_o\sqrt{1-\frac{\gamma^2}{2mk}} = \pm\omega_o\sqrt{S}$

Maximum amplitude is

$R(\pm\omega_o\sqrt{S}) = \frac{F_0}{\sqrt{m^2({\omega_o}^2-(\omega_o\sqrt{S})^2)^2+(\gamma(\omega_o\sqrt{S}))^2}} = \frac{F_0}{\sqrt{m^2({\omega_o}^2-\omega_o^2 S)^2+\gamma^2\omega_o^2 S}}$

$= \frac{F_0}{\sqrt{m^2({\omega_o}^2(1-S))^2+\gamma^2\omega_o^2 S}} = \frac{F_0}{\sqrt{m^2{\omega_o}^4(1-S)^2+\gamma^2\omega_o^2 S}} = \frac{F_0}{\omega_o\sqrt{m^2{\omega_o}^2(1-S)^2+\gamma^2 S}}$

$= \frac{F_0}{\omega_o\sqrt{m^2\frac{k}{m}(1-(1-\frac{\gamma^2}{2mk}))^2+\gamma^2 S}} = \frac{F_0}{\omega_o\sqrt{mk(\frac{\gamma^2}{2mk})^2+\gamma^2 S}} = \frac{F_0}{\omega_o\sqrt{\frac{\gamma^4}{4mk}+\gamma^2 S}} = \frac{F_0}{\omega_o\gamma\sqrt{\frac{\gamma^2}{4mk}+S}}$

$= \frac{F_0}{\omega_o\gamma\sqrt{\frac{\gamma^2}{4mk}+1-\frac{\gamma^2}{2mk}}} = \frac{F_0}{\omega_o\gamma\sqrt{\frac{\gamma^2}{4mk}+1-\frac{2\gamma^2}{4mk}}} = \frac{F_0}{\omega_o\gamma\sqrt{1-\frac{\gamma^2}{4mk}}} \sim \frac{F_0}{\omega_o\gamma}(1+\frac{\gamma^2}{8mk})$ for small $\gamma \neq 0$

via Taylor series expansion about $\gamma = 0$

$\gamma = 0$ (No damping): $mu'' + ku = F_0 cos(\omega t)$

Homogeneous solution: $mr^2 + k = 0$ implies $r = i\sqrt{\frac{k}{m}} = \omega_o$

Thus homogeneous solution: $u = c_1 cos(\omega_o t) + c_2 sin(\omega_o t)$

A non-homogeneous solution to $mu'' + ku = F_0 cos(\omega_o t)$ is of the form

$u = t(Acos(\omega_o t) + Bsin(\omega_o t)) = tRcos(\omega_o - \delta)$. Thus amplitude $\to \infty$ as $t \to \infty$