Given: $y' = f(t, y)$, $y(0) = 0$ Eqn (*)

f, $\partial f/\partial y$ continuous $\forall (t, y) \in (-a, a) \times (-b, b)$. Then

$y = \phi(t)$ is a solution to (*) iff

$\phi'(t) = f(t, \phi(t)), \quad \phi(0) = 0$ iff

$\int_0^t \phi'(s)ds = \int_0^t f(s, \phi(s)ds, \quad \phi(0) = 0$ iff

$\phi(t) = \phi(t) - \phi(0) = \int_0^t f(s, \phi(s)ds$

Thus $y = \phi(t)$ is a solution to (*) iff $\phi(t) = \int_0^t f(s, \phi(s)ds$

Construct ϕ using method of successive approximation – also called Picard's iteration method.

Let $\phi_0(t) = 0$ (or the function of your choice)

Let $\phi_1(t) = \int_0^t f(s, \phi_0(s)ds$

Let $\phi_2(t) = \int_0^t f(s, \phi_1(s)ds$

$\vdots$

Let $\phi_{n+1}(t) = \int_0^t f(s, \phi_n(s)ds$

Let $\phi(t) = \lim_{n \to \infty} \phi_n(t)$

Some questions:

1.) Does $\phi_n(t)$ exist for all n?

2.) Does sequence ϕ_n converge?

3.) Is $\phi(t) = \lim_{n\to\infty} \phi_n(t)$ a solution to (*).

4.) Is the solution unique.

Example: $y' = t + 2y$. That is $f(t, y) = t + 2y$

Let $\phi_0(t) = 0$

Let $\phi_1(t) = \int_0^t f(s, 0)ds = \int_0^t (s + 2(0))ds$

$$= \int_0^t s ds = \frac{s^2}{2}\Big|_0^t = \frac{t^2}{2}$$

Let $\phi_2(t) = \int_0^t f(s, \phi_1(s))ds = \int_0^t f(s, \frac{s^2}{2})ds$

$$= \int_0^t (s + 2(\frac{s^2}{2}))ds = \frac{t^2}{2} + \frac{t^3}{3}$$

Let $\phi_3(t) = \int_0^t f(s, \phi_2(s))ds = \int_0^t f(s, \frac{s^2}{2} + \frac{s^3}{3})ds$

$$= \int_0^t (s + 2\frac{s^2}{2} + \frac{s^3}{3}))ds = \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{6}$$

$\vdots$

See class notes.