Section 2.4 example: $\frac{dy}{dt} = \frac{1}{(1-t)(2-y)}$

$F(y,t) = \frac{1}{(1-t)(2-y)}$ is continuous for all $t \neq 1$, $y \neq 2$
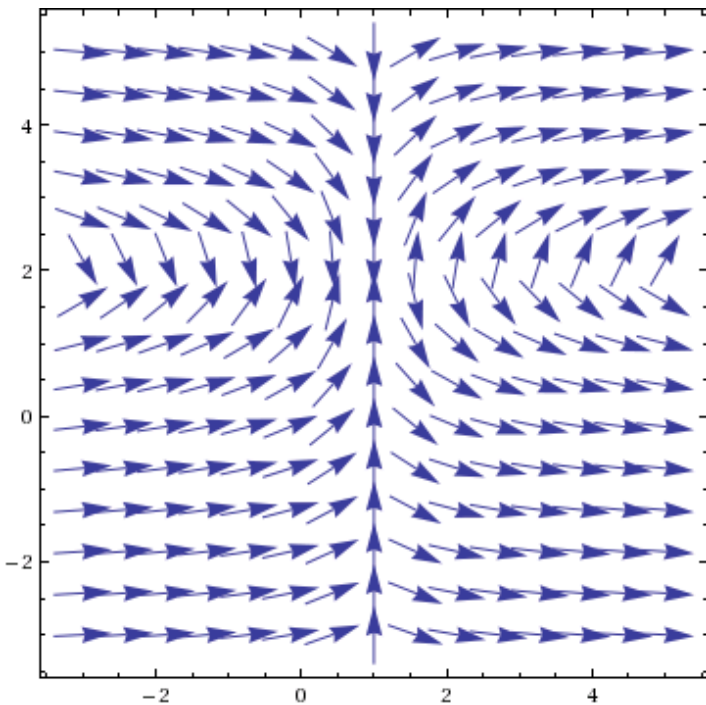
$$\frac{\partial F}{\partial y} = \frac{\partial\left(\frac{1}{(1-t)(2-y)}\right)}{\partial y} = \frac{1}{(1-t)}\frac{\partial(2-y)^{-1}}{\partial y} = \frac{1}{(1-t)(2-y)^2}$$

$\frac{\partial F}{\partial y}$ is continuous for all $t \neq 1$, $y \neq 2$

Thus the IVP $\frac{dy}{dt} = \frac{1}{(1-t)(2-y)}$, $y(t_0) = y_0$ has a unique solution if $t_0 \neq 1$, $y_0 \neq 2$.

Note that if $y_0 = 2$, $\frac{dy}{dt} = \frac{1}{(1-t)(2-y)}$, $y(t_0) = 2$ has two solutions if $t_0 \neq 2$

Note that if $t_0 = 1$, $\frac{dy}{dt} = \frac{1}{(1-t)(2-y)}$, $y(1) = y_0$ has no solutions.

$$(1, 1/((1-x)(2-y)))/sqrt1 + 1/((1-x)(2-y))^2)$$

Solve via separation of variables:

$\int (2-y)dy = \int \frac{dt}{1-t}$

$2y - \frac{y^2}{2} = -ln|1-t| + C$

$y^2 - 4y - 2ln|1-t| + C = 0$

$y = \frac{4 \pm \sqrt{16 + 4(2ln|1-t|+C)}}{2} = 2 \pm \sqrt{4 + 2ln|1-t| + C}$

$$y = 2 \pm \sqrt{2ln|1-t| + C}$$

Find domain: $2ln|1-t| + C \geq 0$

$2ln|1-t| \geq -C$

$ln|1-t| \geq -\frac{C}{2}$ Note: we want to find domain for this C
and thus this C can't swallow constants).

$|1-t| \geq e^{-\frac{C}{2}}$ since e^x is an increasing function.

$1-t \leq -e^{-\frac{C}{2}}$ or $1-t \geq e^{-\frac{C}{2}}$

$-t \leq -e^{-\frac{C}{2}} - 1$ or $-t \geq e^{-\frac{C}{2}} - 1$

Domain: $\begin{cases} t \geq e^{-\frac{C}{2}} + 1 & \text{if } t_0 > 0 \\ t \leq -e^{-\frac{C}{2}} + 1 & \text{if } t_0 < 0. \end{cases}$

Note: Domain is much easier to determine when the ODE is linear.

Find C given $y(t_0) = y_0$: $y_0 = 2 \pm \sqrt{2ln|1 - t_0| + C}$

$\pm(y_0 - 2) = \sqrt{2ln|1 - t_0| + C}$

$(y_0 - 2)^2 - 2ln|1 - t_0| = C$

$y = 2 \pm \sqrt{2ln|1 - t| + C}$

$y = 2 \pm \sqrt{2ln|1 - t| + (y_0 - 2)^2 - 2ln|1 - t_0|}$

$y = 2 \pm \sqrt{(y_0 - 2)^2 + ln\frac{(1-t)^2}{(1-t_0)^2}}$

Domain: $\begin{cases} t \geq e^{-\frac{C}{2}} + 1 & \text{if } t_0 > 0 \\ t \leq -e^{-\frac{C}{2}} + 1 & \text{if } t_0 < 0. \end{cases}$

$e^{-\frac{C}{2}} = e^{-\frac{(y_0 - 2)^2 - 2ln|1-t_0|}{2}} = |1 - t_0|e^{-\frac{(y_0 - 2)^2}{2}}$

Domain: $\begin{cases} t \geq 1 + |1 - t_0|e^{-\frac{(y_0 - 2)^2}{2}} & \text{if } t_0 > 0 \\ t \leq 1 - |1 - t_0|e^{-\frac{(y_0 - 2)^2}{2}} & \text{if } t_0 < 0. \end{cases}$

Section 2.5:

Exponential Growth/Decay
Example: population growth/radioactive decay)

$y' = ry$, $y(0) = y_0$ implies $y = y_0 e^{rt}$

$r > 0$ $\qquad\qquad\qquad\qquad\qquad r < 0$

Logistic growth: $y' = h(y)y$

Example: $y' = r(1 - \frac{y}{K})y$

y vs $f(y)$ slope field:

Equilibrium solutions:

Asymptotically stable:

Asymptotically unstable:

Asymptotically semi-stable:

As $t \to \infty$, if $y > 0$, $y \to$

Solution: $y = \frac{y_0 K}{y_0 + (K - y_0)e^{-rt}}$

Section 2.7 Euler method: Using tangent lines to approximate a function.

$$y_{i+1} = y_i + \Delta y = y_i + \frac{\Delta y}{\Delta t}\Delta t \cong y_i + \frac{dy}{dt}\Delta t$$

Alternatively use equation of tangent line:

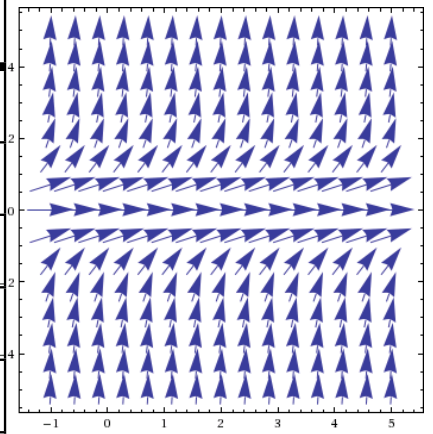$$\text{slope} = \frac{y_{i+1} - y_i}{t_{i+1} - t_i} = f'(y_i, t_i).$$

$$y_{i+1} = f'(y_i, t_i)(t_{i+1} - t_i) + y_i = T(t_{i+1}) \text{ where } y = T(t)$$
$$\text{is the equation of the tangent line at } (y_i, t_i).$$

Example: $\frac{dy}{dt} = y^2$, $y(2) = 1$ implies $y = \frac{1}{3-t}$.

t	$y = 1/(3-t)$	approximation
2.000000	1.000000	1.000000
3.000000	999.000000	2.000000
4.000000	-1.000000	6.000000
5.000000	-0.500000	42.000000
6.000000	-0.333333	1806.000000

t	$y = 1/(3-t)$	approximation
2.000000	1.000000	1.000000
2.100000	1.111111	1.100000
2.200000	1.250000	1.221000
2.300000	1.428571	1.370084
2.400000	1.666667	1.557797
2.500000	2.000000	1.800470
2.600000	2.500000	2.124640
2.700000	3.333333	2.576049
2.800000	5.000000	3.239652
2.900000	10.000004	4.289186

t	$y = 1/(3 - t)$	approximation
2.00	1.000000	1.000000
2.01	1.010101	1.010000
2.02	1.020408	1.020201
2.03	1.030928	1.030609
2.04	1.041667	1.041231
2.05	1.052632	1.052072
2.06	1.063830	1.063141
2.07	1.075269	1.074443
2.08	1.086957	1.085988
2.09	1.098901	1.097782
2.10	1.111111	1.109833
2.11	1.123595	1.122150
2.12	1.136364	1.134742
2.13	1.149425	1.147619
2.87	7.692308	6.721314
2.88	8.333333	7.173075
2.89	9.090908	7.687605
2.90	9.999998	8.278598
2.91	11.111107	8.963949
2.92	12.499993	9.767473
2.93	14.285716	10.721509
2.94	16.666666	11.871017
2.95	19.999996	13.280227
2.96	24.999987	15.043871
2.97	33.333298	17.307051
2.98	49.999897	20.302391
2.99	99.999496	24.424261

$$y' = y^2$$