7.6: Complex eigenvalue example: Solve $\mathbf{x}' = \begin{bmatrix} 3 & -13 \\ 5 & 1 \end{bmatrix} \mathbf{x}$ $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1' \\ x_2' \end{bmatrix}$

Step 1 Find eigenvalues: $det(A - rI) = 0$

$$det(A - rI) = \begin{vmatrix} 3-r & -13 \\ 5 & 1-r \end{vmatrix} = (3-r)(1-r) + 65 = r^2 - 4r + 68 = 0$$

Thus $r = \frac{4 \pm \sqrt{4^2 - 4(68)}}{2} = \frac{4 \pm \sqrt{4(4-68)}}{2} = \frac{4 \pm 2\sqrt{-64}}{2} \boxed{= 2 \pm 8i}$

Step 2 Find eigenvectors: Solve $(A - rI)\mathbf{x} = \mathbf{0}$

$$A - (2 \pm 8i)I = \begin{bmatrix} 3 - (2 \pm 8i) & -13 \\ 5 & 1 - (2 \pm 8i) \end{bmatrix} = \begin{bmatrix} 1 \mp 8i & -13 \\ 5 & -1 \mp 8i \end{bmatrix}$$

$\lambda = 2 + 8i$

$\downarrow$

$\begin{bmatrix} 1 \\ 5 \end{bmatrix} + i \begin{bmatrix} 8 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 13 \\ 1 \end{bmatrix} + i \begin{bmatrix} 0 \\ -8 \end{bmatrix}$

Solve $\begin{bmatrix} 1 \mp 8i & -13 \\ 5 & -1 \mp 8i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 1 \mp 8i & -13 \\ 5 & -1 \mp 8i \end{bmatrix} \begin{bmatrix} 13 \\ 1 \mp 8i \end{bmatrix} = \begin{bmatrix} (1 \mp 8i)13 - 13(1 \mp 8i) \\ 5(13) + (-1 \mp 8i)(1 \mp 8i) \end{bmatrix} = \begin{bmatrix} 0 \\ 65 + (-1 + 64i^2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Thus any non-zero multiple of $\begin{bmatrix} 13 \\ 1 \mp 8i \end{bmatrix}$ is an eigenvector of A with eigen value $2 \pm 8i$.

Note: $\begin{bmatrix} 1 \pm 8i \\ 5 \end{bmatrix}$ is a multiple of $\begin{bmatrix} 13 \\ 1 \mp 8i \end{bmatrix}$ since $\begin{bmatrix} 1 \mp 8i & -13 \\ 5 & -1 \mp 8i \end{bmatrix} \begin{bmatrix} 1 \pm 8i \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Thus we can use either $\begin{bmatrix} 1 \pm 8i \\ 5 \end{bmatrix} = \boxed{\begin{bmatrix} 1 \\ 5 \end{bmatrix} \pm i \begin{bmatrix} 8 \\ 0 \end{bmatrix}}$ or $\begin{bmatrix} 13 \\ 1 \mp 8i \end{bmatrix}$ or any nonzero multiple.

$\boxed{2}+\boxed{8}i$

General solution:

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 e^{2t} \left(\begin{bmatrix} 1 \\ 5 \end{bmatrix} cos(8t) - \begin{bmatrix} 8 \\ 0 \end{bmatrix} sin(8t) \right) + c_2 e^{2t} \left(\begin{bmatrix} 1 \\ 5 \end{bmatrix} sin(8t) + \begin{bmatrix} 8 \\ 0 \end{bmatrix} cos(8t) \right)}$$

Slope field for x_2 vs x_1: $\frac{dx_2}{dx_1} = \frac{\frac{dx_2}{dt}}{\frac{dx_1}{dt}} = \frac{x_2'}{x_1'} = \frac{3x_1 - 13x_2}{5x_1 + x_2}$

Note slope 0's occur when $3x_1 - 13x_2 = 0$, ie, $x_2 = \frac{13}{3}x_1$.

Note slope ∞'s occur when $5x_1 + x_2 = 0$, ie, $x_2 = -5x_1$.

Determine where slopes are positive vs negative for regions between these lines.

For example, along the x_2 axis slope is negative: $x_1 = 0$ and $\frac{dx_2}{dx_1} = \frac{-13x_2}{x_2} = -13$

For example, along the x_1 axis slope is positive: $x_2 = 0$ and $\frac{dx_2}{dx_1} = \frac{3x_1}{5x_1} = \frac{3}{5}$

Equilibrium soln

$\vec{x}' = A\vec{x}$

constant soln

Ch 7 and 9 $\vec{x}' = 0 \ \forall t \Rightarrow \vec{x} = 0$

$x_1(t) = 0$

$x_2(t) = 0$

Suppose an object moves in the 2D plane (the x_1, x_2 plane) so that it is at the point $(x_1(t), x_2(t))$ at time t. Suppose the object's velocity is given by

$$x_1'(t) = ax_1 + bx_2,$$
$$x_2'(t) = cx_1 + dx_2$$

Or in matrix form $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

To solve, find eigenvalues and corresponding eigenvectors:

$$\begin{vmatrix} a-r & b \\ c & d-r \end{vmatrix} = (a-r)(d-r) - bc = r^2 - (a+d)r + ad - bc = 0.$$

Thus $r = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$

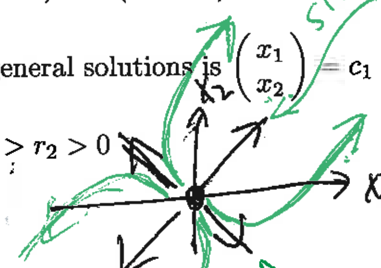
Case 1: $(a+d)^2 - 4(ad-bc) > 0$

Hence the general solutions is $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} e^{r_1 t} + c_2 \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} e^{r_2 t}$

$r_1 > r_2 > 0$

slope $\frac{v_2}{v_1}$

Case 1a: $r_1 > r_2 > 0$



unstable

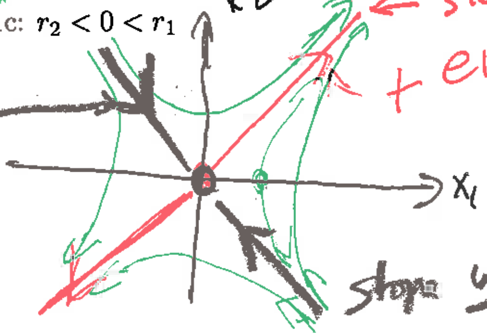
r_2 e.value $< r_1$

slope $\frac{w_2}{w_1}$

Case 1b: $r_1 < r_2 < 0$

stable

Case 1c: $r_2 < 0 < r_1$



- e. value

slope $\frac{v_2}{v_1}$

+ eigenvalue

$c_2 = 0$

$x_1 = c_1 v_1 e^{r_1 t}$

$x_2 = c_1 v_2 e^{r_1 t}$

slope $\frac{w_2}{w_1}$

$x_2 = \frac{v_2}{v_1} x_1$

unstable saddle pt

Case 2: $(a+d)^2 - 4(ad-bc) = 0$

Case 2i: Two independent eigenvectors:

The general solution is $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} e^{rt} + c_2 \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} e^{rt}$

Case 2ii: One independent eigenvectors:

The general solution is $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} e^{rt} + c_2 \left[\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} t + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \right] e^{rt}$

Case 2a: $r > 0$ Case 2b: $r < 0$

$r > 0$

$r < 0$

Case 3: $(a+d)^2 - 4(ad-bc) < 0$. I.e., $r = \lambda \pm i\mu$

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Suppose eigenvector corresponding to eigenvalue is

$$\begin{pmatrix} v_1 \pm iw_1 \\ v_2 \pm iw_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \pm i \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

Then general solution is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 e^{\lambda t} \left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} cos(\mu t) - \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} sin(\mu t) \right)$$
$$+ c_2 e^{\lambda t} \left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} sin(\mu t) + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} cos(\mu t) \right)$$

Case 3a: $\lambda > 0$

Case 3a: $\lambda < 0$

Case 3a: $\lambda = 0$

$X' = Ax$ if $x' = 0$

$\vec{X}(t) = c = \begin{bmatrix}0\\0\end{bmatrix}$

Suppose an object moves in the 2D plane (the x_1, x_2 plane) so that it is at the point $(x_1(t), x_2(t))$ at time t. Suppose the object's velocity is given by

$$x_1'(t) = ax_1 + bx_2,$$
$$x_2'(t) = cx_1 + dx_2$$

Or in matrix form $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

To solve, find eigenvalues and corresponding eigenvectors:

$$\begin{vmatrix} a-r & b \\ c & d-r \end{vmatrix} = (a-r)(d-r) - bc = r^2 - (a+d)r + ad - bc = 0.$$

$$\text{Thus } r = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$$

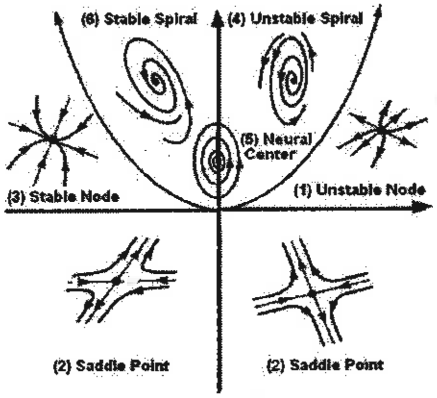
Let $p = trace(A) = a + d$ and let $q = det A = ad - bc$

Then $r = \frac{p \pm \sqrt{p^2-4q}}{2}$ 2 real roots $\Rightarrow p^2 - 4q > 0$:

$\Rightarrow p^2 > 4q$

$q < \frac{p^2}{4}$

Thus the type of solution depends on (p, q)



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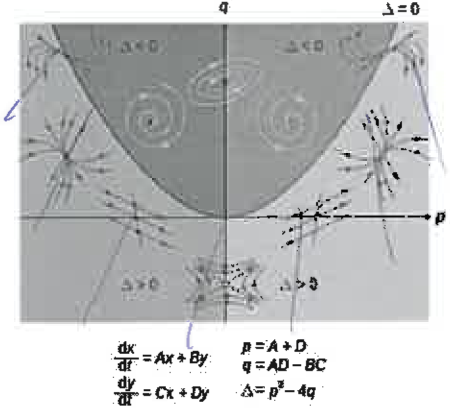


$\frac{dx}{dt} = Ax + By$

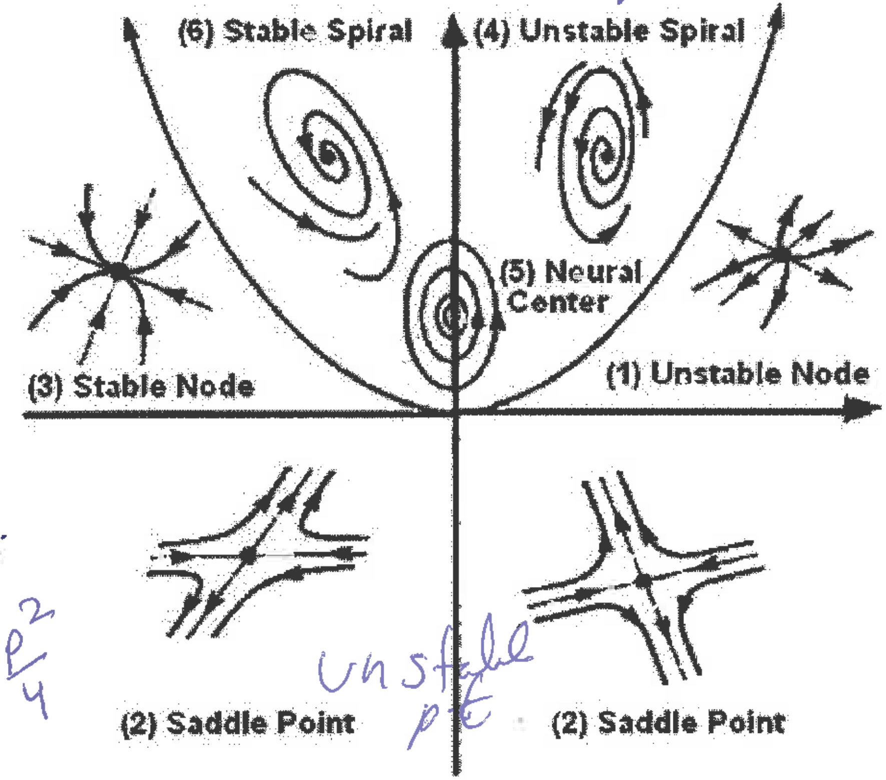
$\frac{dy}{dt} = Cx + Dy$

$p = A + D$

$q = AD - BC$

$\Delta = p^2 - 4q$

https://en.wikipedia.org/wiki/Phase_plane



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$q = \frac{p^2}{4}$