MATH:7450 (22M:305) Topics in Topology: Scientific and Engineering Applications of Algebraic Topology

Sept 30, 2013: Multidimensional Persistence.

Fall 2013 course offered through the
University of Iowa Division of Continuing Education

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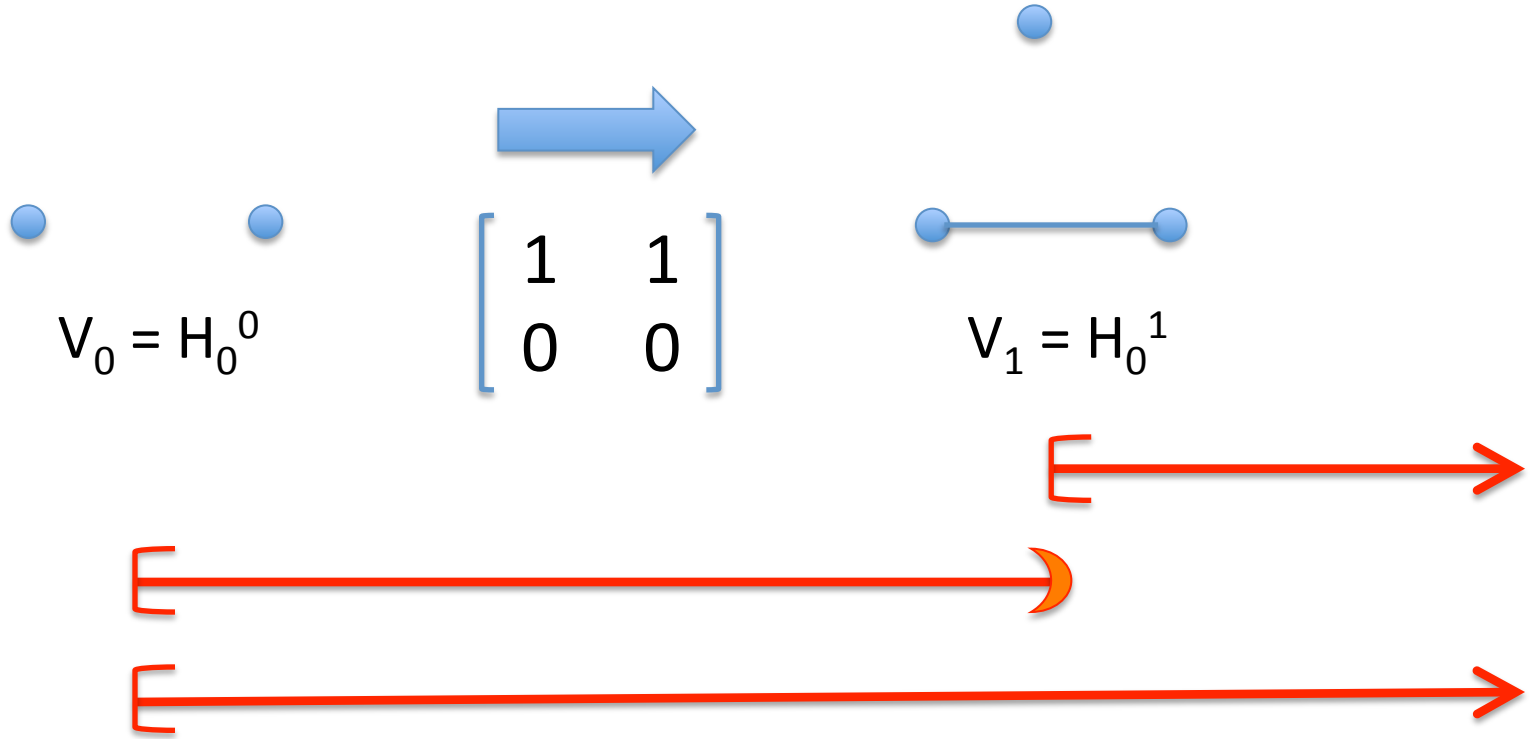
Applied Mathematical and Computational Sciences,
University of Iowa

http://www.math.uiowa.edu/~idarcy/AppliedTopology.html

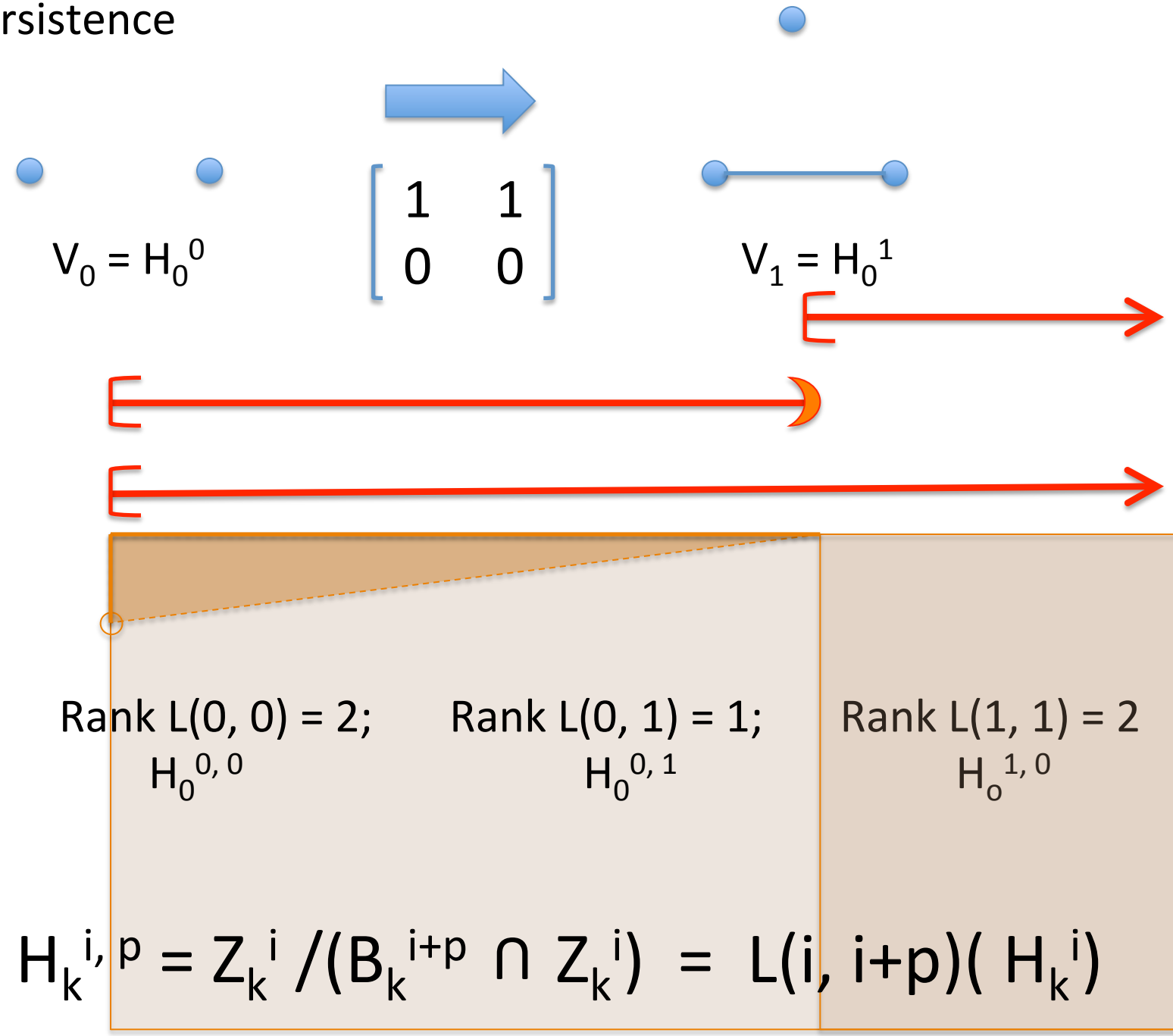
1D persistence



Rank L(0, 0) = 2; Rank L(0, 1) = 1; Rank L(1, 1) = 2

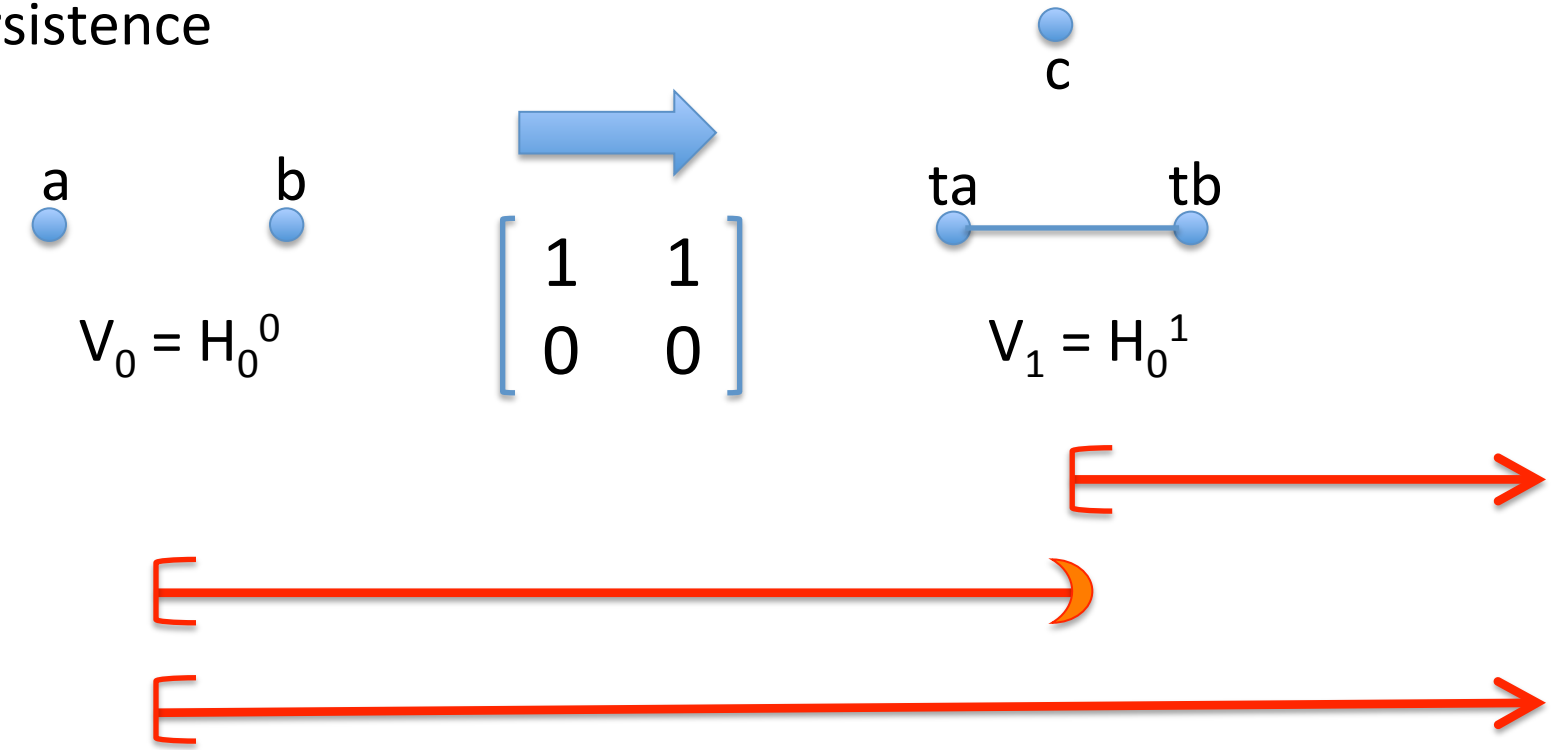
Note Rank determines persistence bars and vice versa

1D persistence



$$H_k^{i,p} = Z_k^i / (B_k^{i+p} \cap Z_k^i) = L(i, i+p)(H_k^i)$$

1D persistence



$$\begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} ab \\ \left[\begin{array}{c} -t \\ t \\ 0 \end{array}\right] \end{array}$$

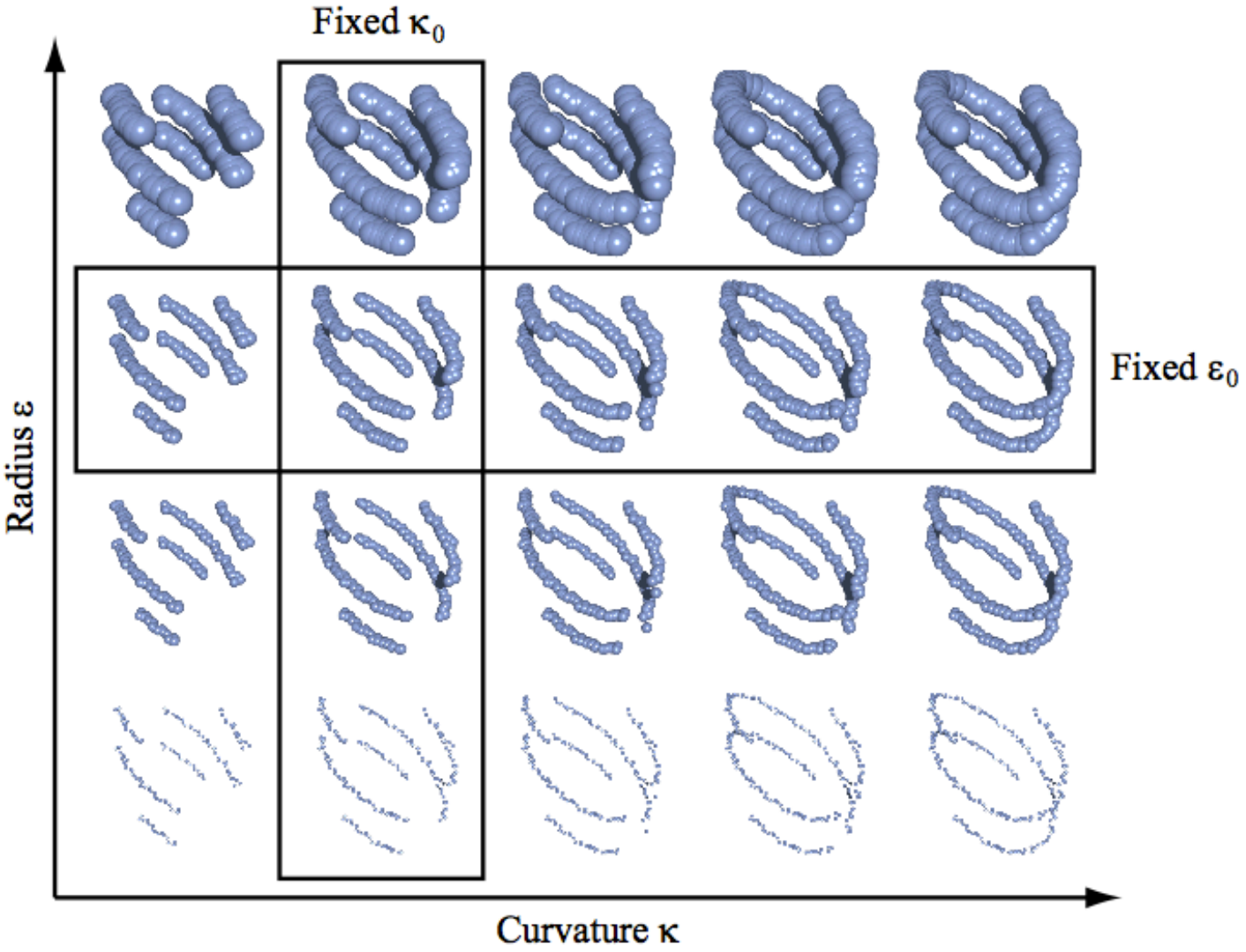
$\partial(ab) = t(b - a)$

H_0 Persistence module: $F[t] \oplus F[t]/(t) \oplus \Sigma^1 F[t]$

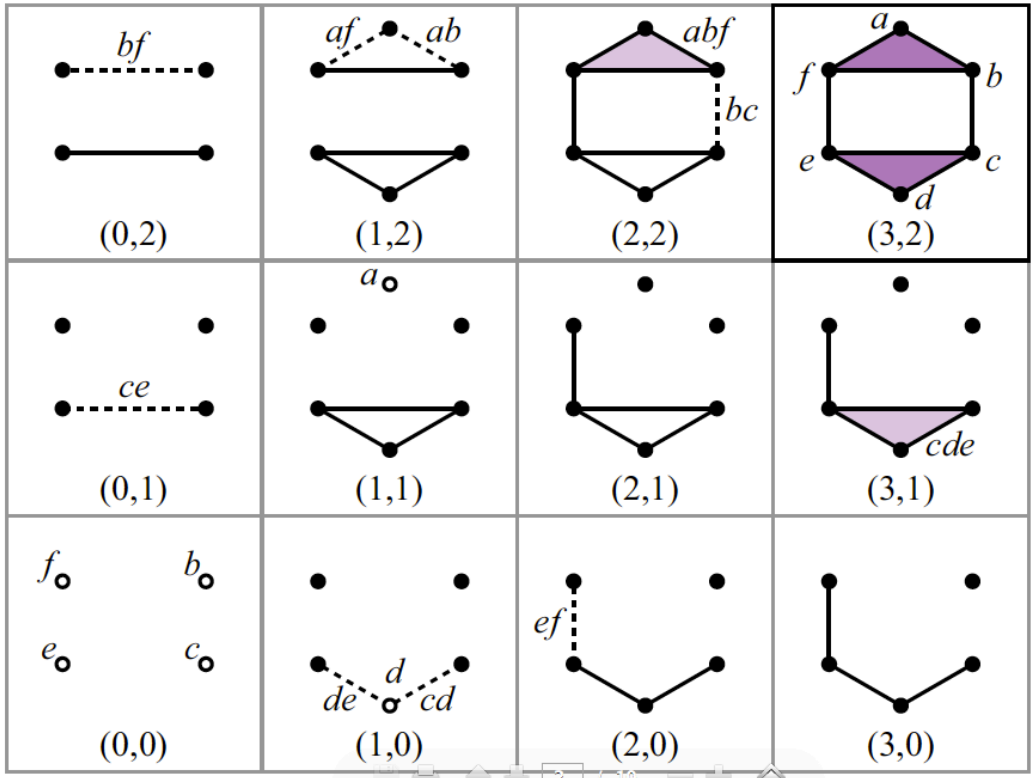
$$(\ \Sigma\, f_n t^n, \quad f, \quad t\, \Sigma\, f_n t^n\ )$$

The Theory of Multidimensional Persistence, Gunnar Carlsson, Afra Zomorodian

"Persistence and Point Clouds" Functoriality, diagrams, difficulties in classifying diagrams, multidimensional persistence, Gröbner bases, Gunnar Carlsson
http://www.ima.umn.edu/videos/?id=862



		ab	bc	cd	de	ef	af	bf	ce
		$x_1x_2^2$	$x_1^2x_2^2$	x_1	x_1	x_1^2	$x_1x_2^2$	x_2^2	x_2
a	x_1x_2	x_2	0	0	0	0	x_2	0	0
d	x_1	0	0	1	1	0	0	0	0
b	1	$x_1x_2^2$	$x_1^2x_2^2$	0	0	0	0	x_2^2	0
c	1	0	$x_1^2x_2^2$	x_1	0	0	0	0	x_2
e	1	0	0	0	x_1	x_1^2	0	0	x_2
f	1	0	0	0	0	x_1^2	$x_1x_2^2$	x_2^2	0



Computing Multidimensional Persistence, Gunnar Carlsson, Gurjeet Singh, and Afra Zomorodian

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\begin{array}{c}
 & ab & bc & cd & de & ef & af & bf & ce \\
a & x_2 & 0 & 0 & 0 & 0 & x_2 & 0 & 0 \\
d & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
b & x_1x_2^2 & x_1^2x_2^2 & 0 & 0 & 0 & 0 & x_2^2 & 0 \\
c & 0 & x_1^2x_2^2 & x_1 & 0 & 0 & 0 & 0 & x_2 \\
e & 0 & 0 & 0 & x_1 & x_1^2 & 0 & 0 & x_2 \\
f & 0 & 0 & 0 & 0 & x_1^2 & x_1x_2^2 & x_2^2 & 0
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a & x_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
d & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
b & x_1x_2^2 & x_1^2x_2^2 & 0 & 0 & 0 & x_1x_2^2 & x_2^2 & 0 \\
c & 0 & x_1^2x_2^2 & x_1 & 0 & 0 & 0 & 0 & x_2 \\
e & 0 & 0 & 0 & x_1 & x_1^2 & 0 & 0 & x_2 \\
f & 0 & 0 & 0 & 0 & x_1^2 & x_1x_2^2 & x_2^2 & 0
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a & x_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
d & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
b & x_1x_2^2 & x_1^2x_2^2 & 0 & 0 & 0 & x_1x_2^2 & x_2^2 & 0 \\
c & 0 & x_1^2x_2^2 & x_1 & 0 & 0 & 0 & 0 & x_2 \\
e & 0 & 0 & 0 & x_1 & x_1^2 & 0 & 0 & x_2 \\
f & 0 & 0 & 0 & 0 & x_1^2 & x_1x_2^2 & x_2^2 & 0
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a & x_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
d & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
b & x_1x_2^2 & x_1^2x_2^2 & 0 & 0 & 0 & x_1x_2^2 & x_2^2 & 0 \\
c & 0 & x_1^2x_2^2 & x_1 & x_1 & 0 & 0 & 0 & x_2 \\
e & 0 & 0 & 0 & x_1 & x_1^2 & 0 & 0 & x_2 \\
f & 0 & 0 & 0 & 0 & x_1^2 & x_1x_2^2 & x_2^2 & 0
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d & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
b & x_1x_2^2 & x_1^2x_2^2 & 0 & 0 & 0 & x_1x_2^2 & x_2^2 & 0 \\
c & 0 & x_1^2x_2^2 & x_1 & x_1 & 0 & 0 & 0 & x_2 \\
e & 0 & 0 & 0 & x_1 & x_1^2 & 0 & 0 & x_2 \\
f & 0 & 0 & 0 & 0 & x_1^2 & x_1x_2^2 & x_2^2 & 0
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a & x_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
d & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
b & x_1x_2^2 & 0 & x_1^2x_2^2 & 0 & 0 & x_1x_2^2 & x_2^2 & 0 \\
c & 0 & x_1 & x_1^2x_2^2 & x_1 & 0 & 0 & 0 & x_2 \\
e & 0 & 0 & 0 & x_1 & x_1^2 & 0 & 0 & x_2 \\
f & 0 & 0 & 0 & 0 & x_1^2 & x_1x_2^2 & x_2^2 & 0
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x_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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0 & x_1 & x_1^2x_2^2 & x_1 & 0 & 0 & 0 & x_2 \\
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0 & x_1 & 0 & x_1^2x_2^2 & x_1 & 0 & 0 & x_2 \\
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\end{array}\right)
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x_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
x_1x_2^2 & 0 & x_2^2 & x_1^2x_2^2 & 0 & 0 & x_1x_2^2 & 0 \\
0 & x_1 & 0 & x_1^2x_2^2 & x_1 & 0 & 0 & x_2 \\
0 & 0 & 0 & 0 & x_1 & x_1^2 & 0 & x_2 \\
0 & 0 & x_2^2 & 0 & 0 & x_1^2 & x_1x_2^2 & 0
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x_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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x_1x_2^2 & 0 & x_2^2 & 0 & 0 & 0 & x_1x_2^2 & 0 \\
0 & x_1 & 0 & x_1^2x_2^2 & x_1 & 0 & 0 & x_2 \\
0 & 0 & 0 & 0 & x_1 & x_1^2 & 0 & x_2 \\
0 & 0 & x_2^2 & x_1^2x_2^2 & 0 & x_1^2 & x_1x_2^2 & 0
\end{array}\right)
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x_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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0 & x_1 & 0 & x_1^2 x_2^2 & x_1 & 0 & 0 & x_2 \\
0 & 0 & 0 & 0 & x_1 & x_1^2 & 0 & x_2 \\
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x_1 x_2^2 & 0 & x_2^2 & 0 & 0 & 0 & 0 & 0 \\
0 & x_1 & 0 & x_1^2 x_2^2 & x_1 & 0 & 0 & x_2 \\
0 & 0 & 0 & 0 & x_1 & x_1^2 & 0 & x_2 \\
0 & 0 & x_2^2 & x_1^2 x_2^2 & 0 & x_1^2 & 0 & 0
\end{array}\right)
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x_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
x_1x_2^2 & 0 & x_2^2 & 0 & 0 & 0 & 0 & 0 \\
0 & x_1 & 0 & x_1^2x_2^2 & x_1 & 0 & 0 & x_2 \\
0 & 0 & 0 & 0 & x_1 & x_1^2 & 0 & x_2 \\
0 & 0 & x_2^2 & x_1^2x_2^2 & 0 & x_1^2 & 0 & 0
\end{array} \right)
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0 & x_1 & 0 & x_1 & x_1^2x_2^2 & 0 & 0 & x_2 \\
0 & 0 & 0 & x_1 & 0 & x_1^2 & 0 & x_2 \\
0 & 0 & x_2^2 & 0 & x_1^2x_2^2 & x_1^2 & 0 & 0
\end{array} \right)
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x_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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x_1x_2^2 & 0 & x_2^2 & 0 & 0 & 0 & 0 & 0 \\
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0 & 0 & 0 & x_1 & 0 & x_1^2 & 0 & x_2 \\
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\end{array}\right)
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x_1x_2^2 & 0 & x_2^2 & 0 & 0 & 0 & 0 & 0 \\
0 & x_1 & 0 & x_1 & 0 & 0 & 0 & x_2 \\
0 & 0 & 0 & x_1 & x_1^2x_2^2 & x_1^2 & 0 & x_2 \\
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\end{array}\right)
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\left(\begin{array}{ccccccccc}
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x_1x_2^2 & 0 & x_2^2 & 0 & 0 & 0 & 0 & 0 \\
0 & x_1 & 0 & x_1 & 0 & 0 & 0 & x_2 \\
0 & 0 & 0 & x_1 & x_1^2x_2^2 & x_1^2 & 0 & x_2 \\
0 & 0 & x_2^2 & 0 & x_1^2x_2^2 & x_1^2 & 0 & 0
\end{array} \right)
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x_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
x_1x_2^2 & 0 & x_2^2 & 0 & 0 & 0 & 0 & 0 \\
0 & x_1 & 0 & x_1 & 0 & 0 & 0 & x_2 \\
0 & 0 & 0 & x_1 & x_1^2 & x_1^2x_2^2 & 0 & x_2 \\
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\end{array} \right)
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a \\
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ab & cd & bf & de+cd & ef & bc+x_1^2bf+x_1x_2^2(de+cd) & af+ab+x_1bf & ce
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x_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
x_1x_2^2 & 0 & x_2^2 & 0 & 0 & 0 & 0 & 0 \\
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\end{array}\right)
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ab & cd & bf & de+cd & ef & bc+x_1^2bf+x_1x_2^2(de+cd)+x_2^2ef & af+ab+x_1bf & ce
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x_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
x_1x_2^2 & 0 & x_2^2 & 0 & 0 & 0 & 0 & 0 \\
0 & x_1 & 0 & x_1 & 0 & 0 & 0 & x_2 \\
0 & 0 & 0 & x_1 & x_1^2 & 0 & 0 & x_2 \\
0 & 0 & x_2^2 & 0 & x_1^2 & 0 & 0 & 0
\end{array}\right)
\end{array}
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\begin{array}{c}
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a^{(1,1)} \\
d^{(1,0)} \\
b^{(0,0)} \\
c^{(0,0)} \\
e^{(0,0)} \\
f^{(0,0)}
\end{array}
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ab & cd & bf & de+cd & ce & ef & bc+x_1^2bf+x_1x_2^2(de+cd)+x_2^2ef & af+ab+x_1bf \\
\hline
x_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
x_1x_2^2 & 0 & x_2^2 & 0 & 0 & 0 & 0 & 0 \\
0 & x_1 & 0 & x_1 & x_2 & 0 & 0 & 0 \\
0 & 0 & 0 & x_1 & x_2 & x_1^2 & 0 & 0 \\
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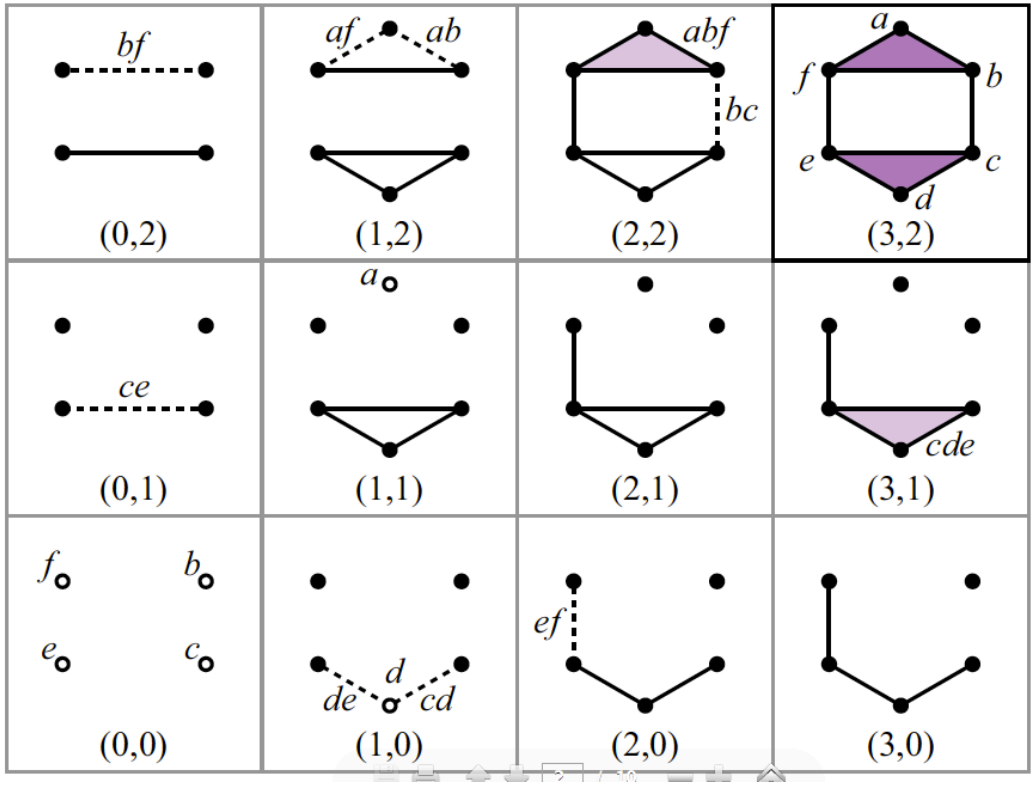
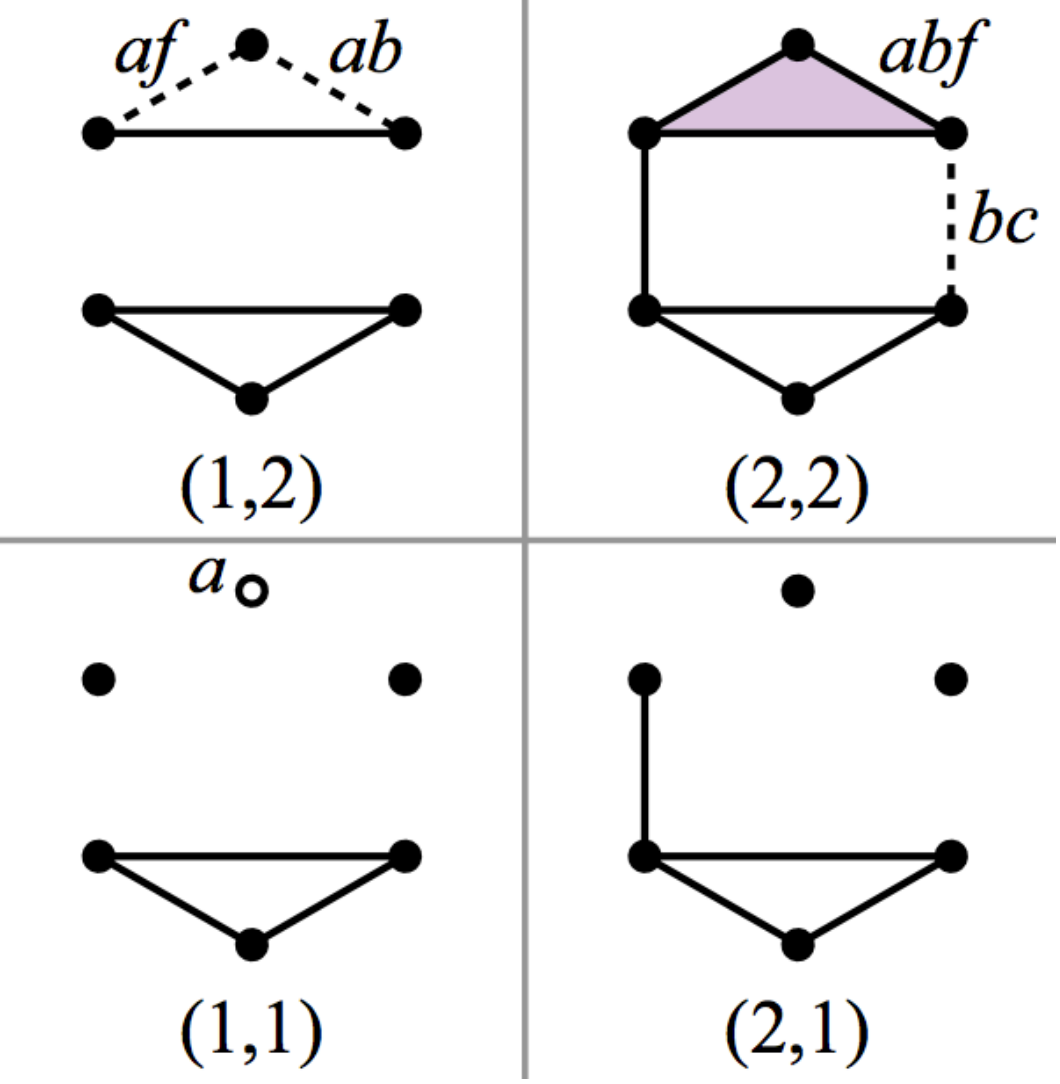
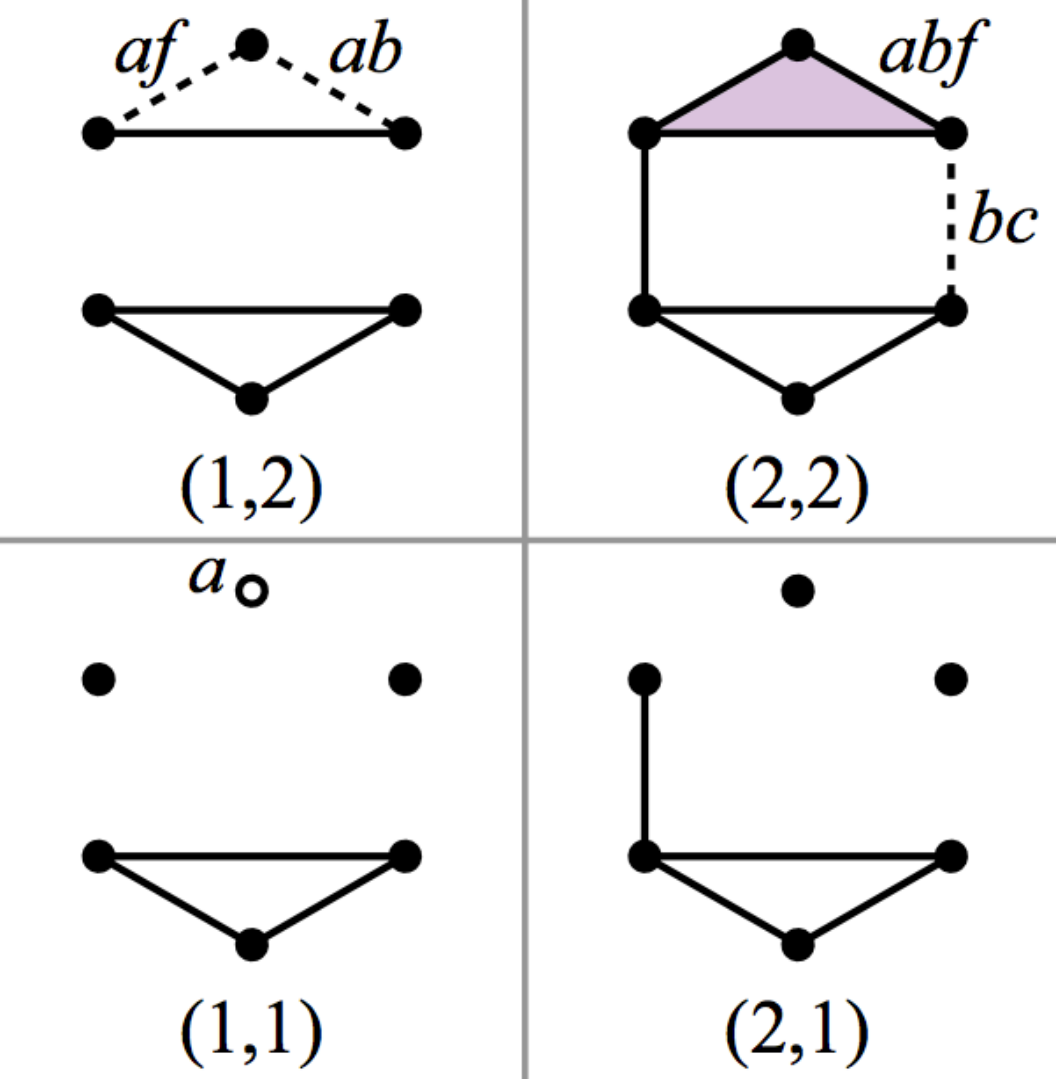




$$ab$$

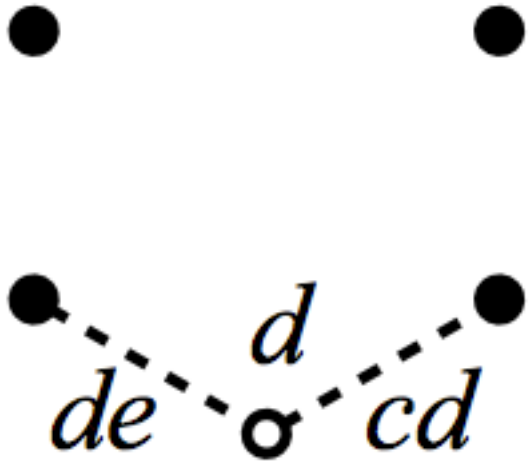
$$a^{(1,1)} \quad \begin{pmatrix} x_2 \end{pmatrix}$$



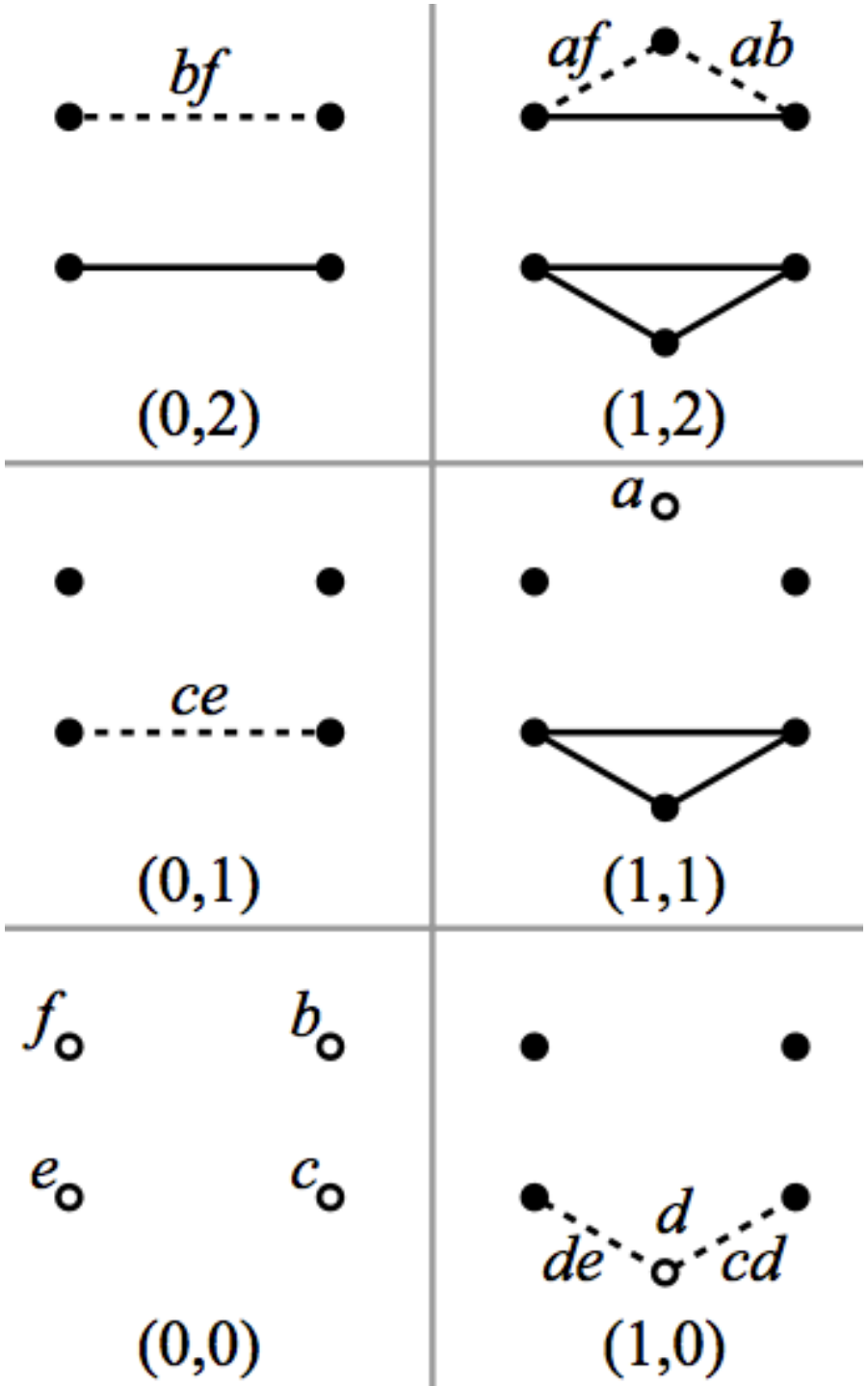
$\Sigma^{(1,\,1)}\, Z_2[x_1, x_2]/x_2$

$$ab$$

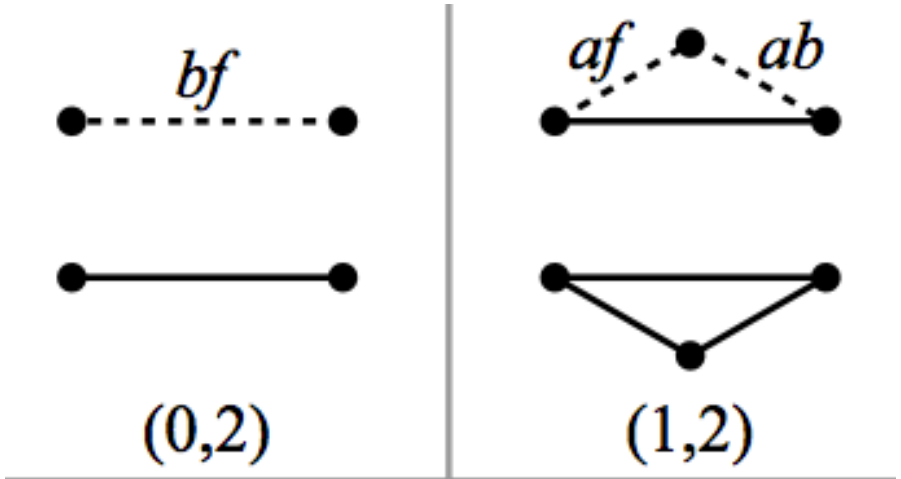
$$a^{(1,1)} \left(\quad x_2 \right.$$

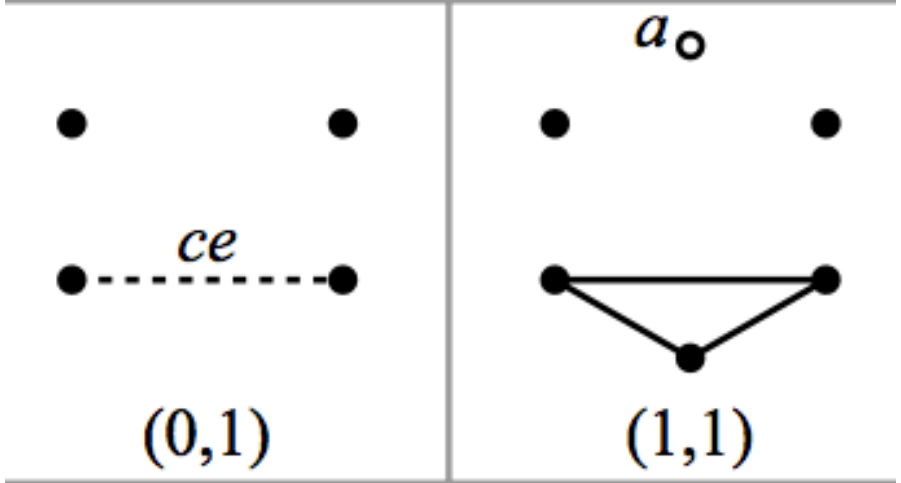


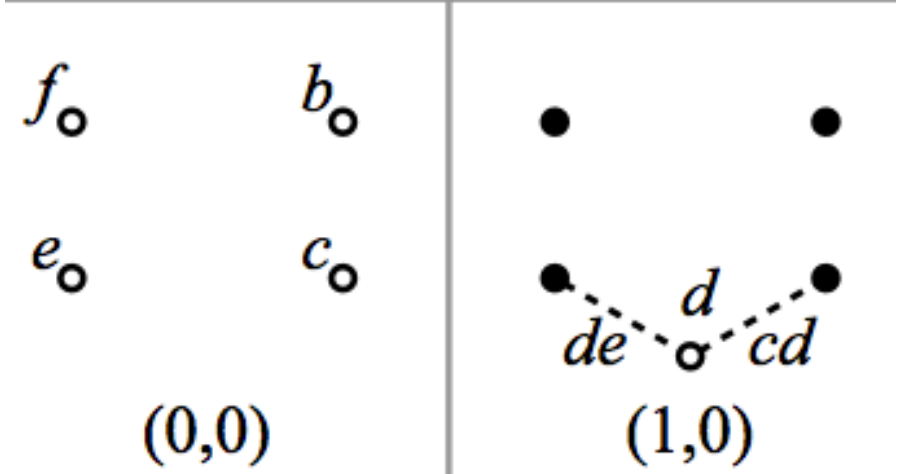
$$
\begin{array}{c} \\ a^{(1,1)} \\ d^{(1,0)} \end{array}
\begin{array}{c} ab \\ \begin{pmatrix} x_2 \\ 0 \end{pmatrix} \end{array}
\begin{array}{c} cd \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{array}
$$



$$
\begin{array}{c} \\ a^{(1,1)} \\ d^{(1,0)} \\ b^{(0,0)} \\ \\ \end{array}
\begin{array}{c}
\begin{array}{ccc} ab & cd & bf \end{array} \\
\left(\begin{array}{ccc} x_2 & 0 & 0 \\ 0 & 1 & 0 \\ x_1x_2^2 & 0 & x_2^2 \end{array}\right.
\end{array}
$$



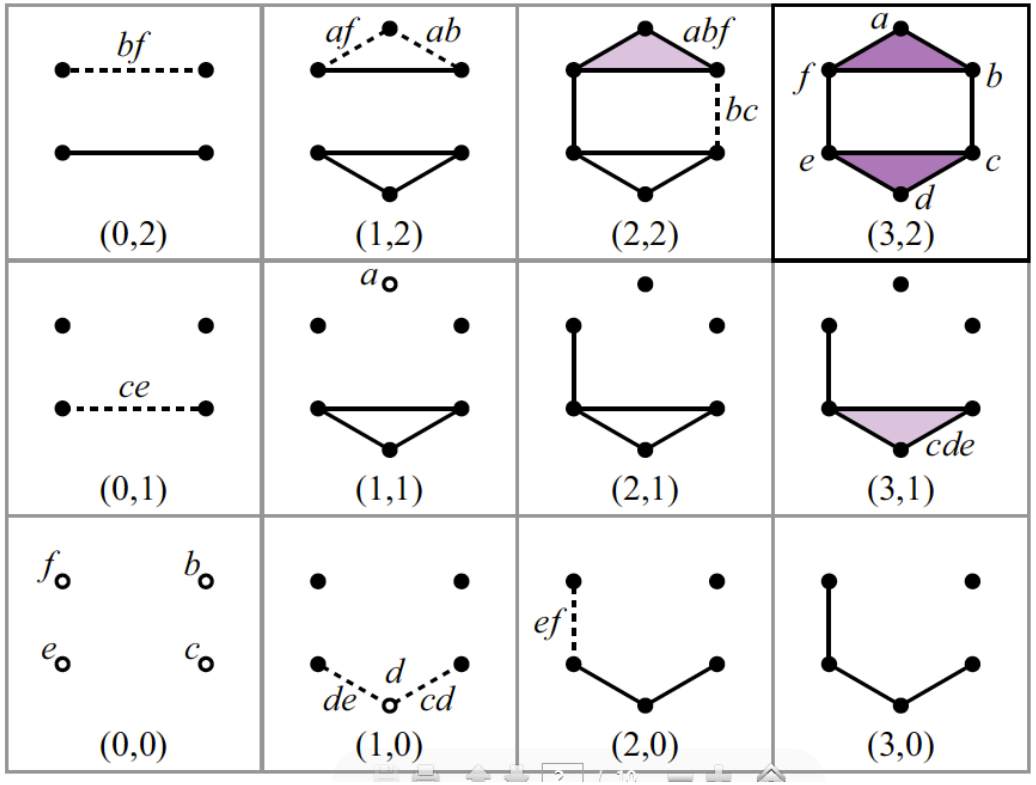


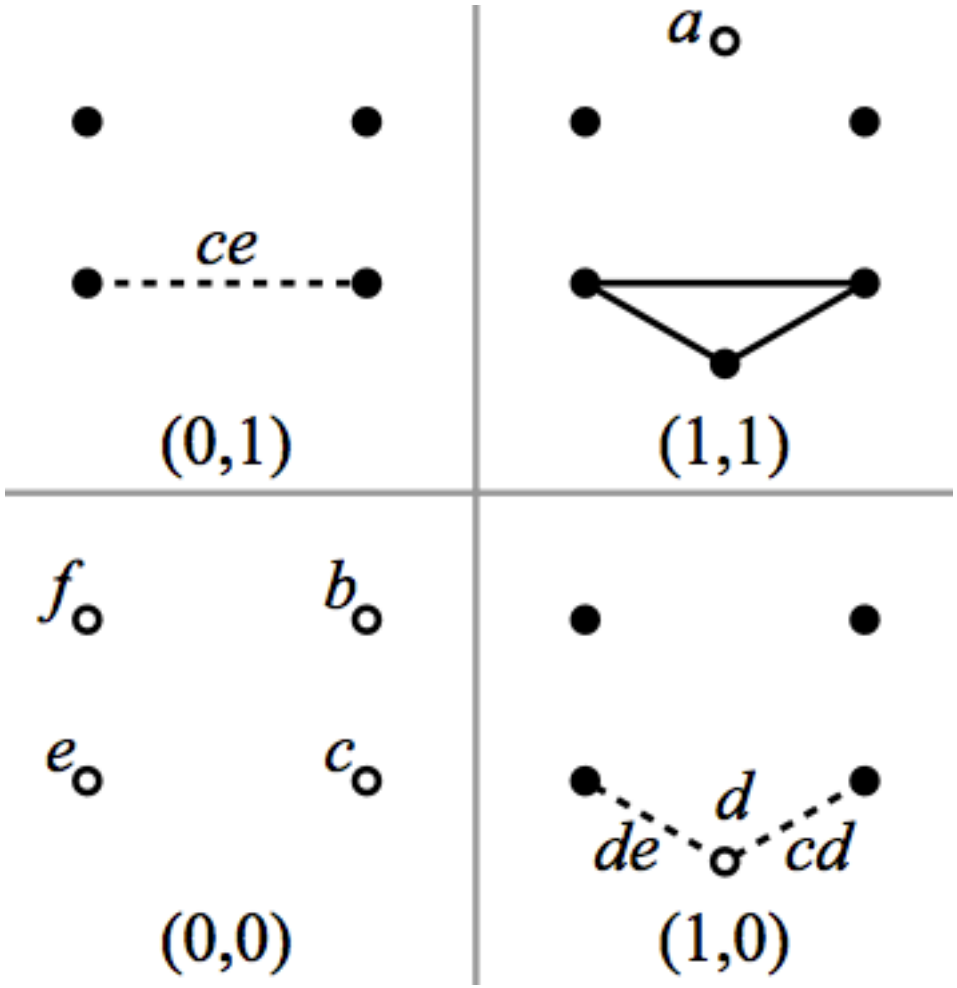


$Z_2[x_1, x_2]/x_2^2$

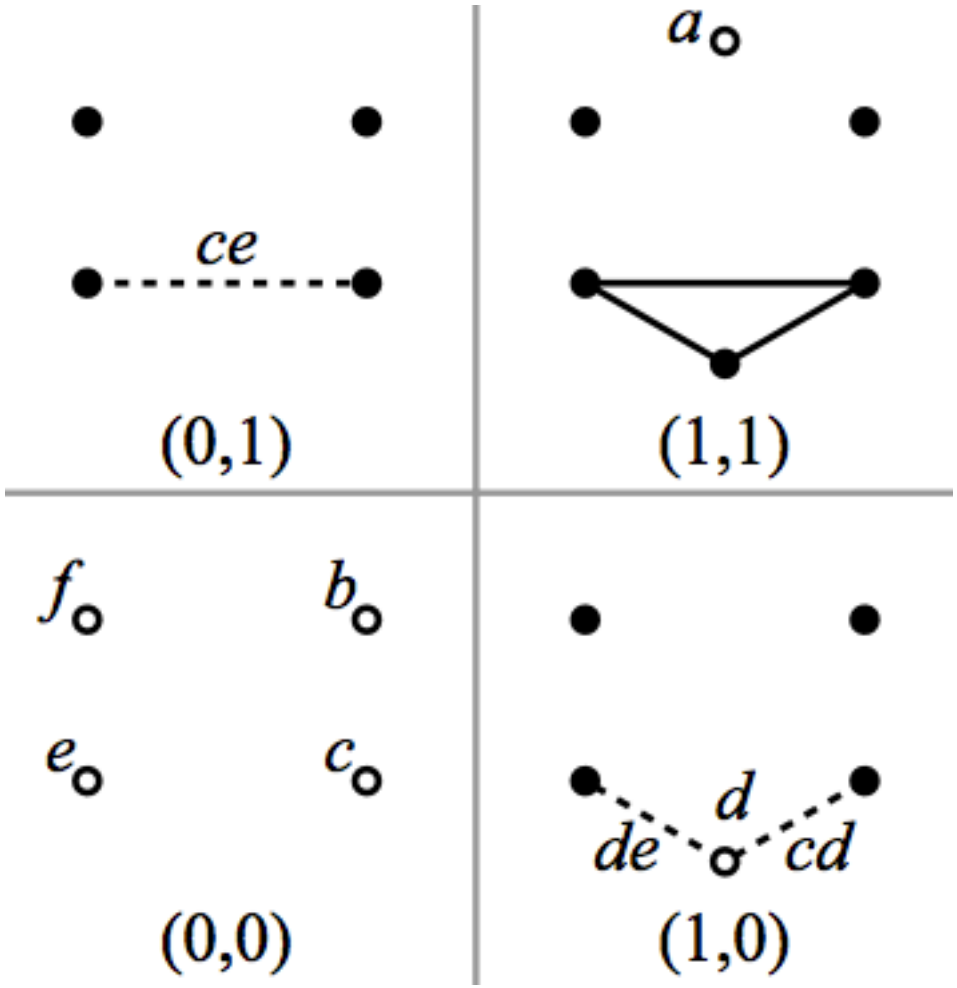
$$\begin{array}{c} \\ a^{(1,1)} \\ d^{(1,0)} \\ b^{(0,0)} \end{array} \begin{array}{c} \begin{array}{ccc} ab & cd & bf \end{array} \\ \begin{pmatrix} x_2 & 0 & 0 \\ 0 & 1 & 0 \\ x_1 x_2^2 & 0 & x_2^2 \end{pmatrix} \end{array}$$

$$
\begin{array}{c}
 \\
a^{(1,1)} \\
d^{(1,0)} \\
b^{(0,0)} \\
c^{(0,0)} \\
e^{(0,0)} \\
f^{(0,0)}
\end{array}
\begin{array}{c}
\begin{array}{cccccc|cc}
ab & cd & bf & de+cd & ce & ef & bc+x_1^2bf+x_1x_2^2(de+cd)+x_2^2ef & af+ab+x_1bf \\
\hline
x_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
x_1x_2^2 & 0 & x_2^2 & 0 & 0 & 0 & 0 & 0 \\
0 & x_1 & 0 & x_1 & x_2 & 0 & 0 & 0 \\
0 & 0 & 0 & x_1 & x_2 & x_1^2 & 0 & 0 \\
0 & 0 & x_2^2 & 0 & 0 & x_1^2 & 0 & 0
\end{array}
\end{array}
$$





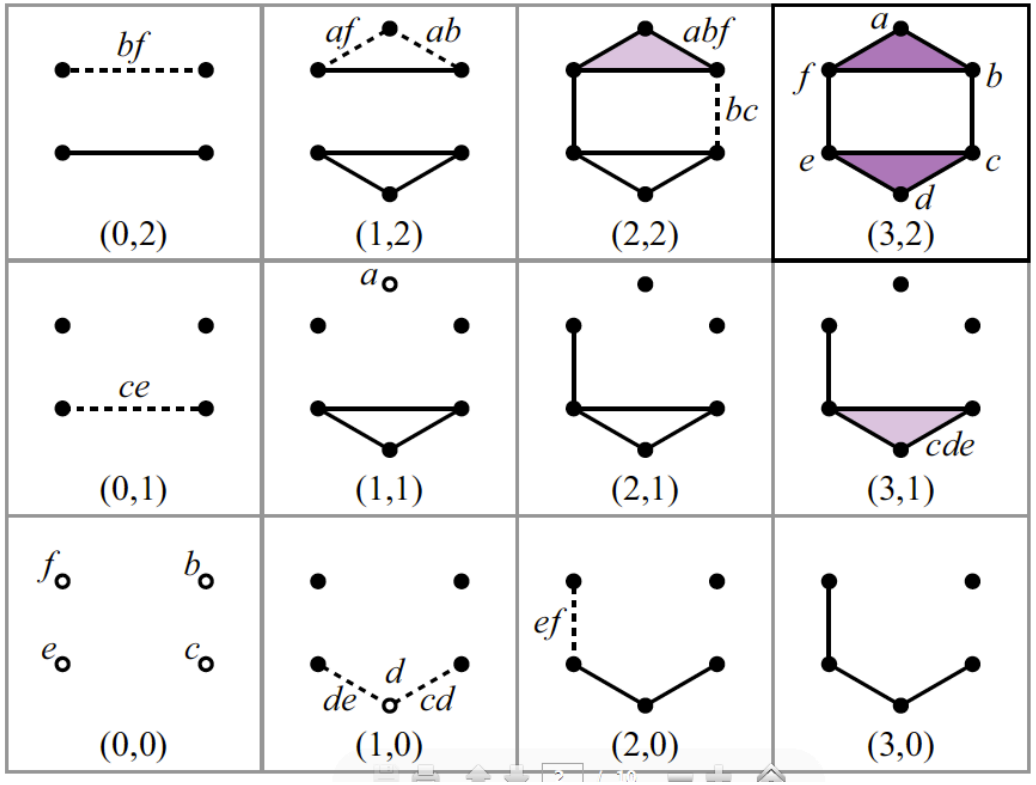
$$c^{(0,0)} \quad | \quad 0 \quad x_1 \quad 0 \quad x_1 \quad x_2 \quad 0$$

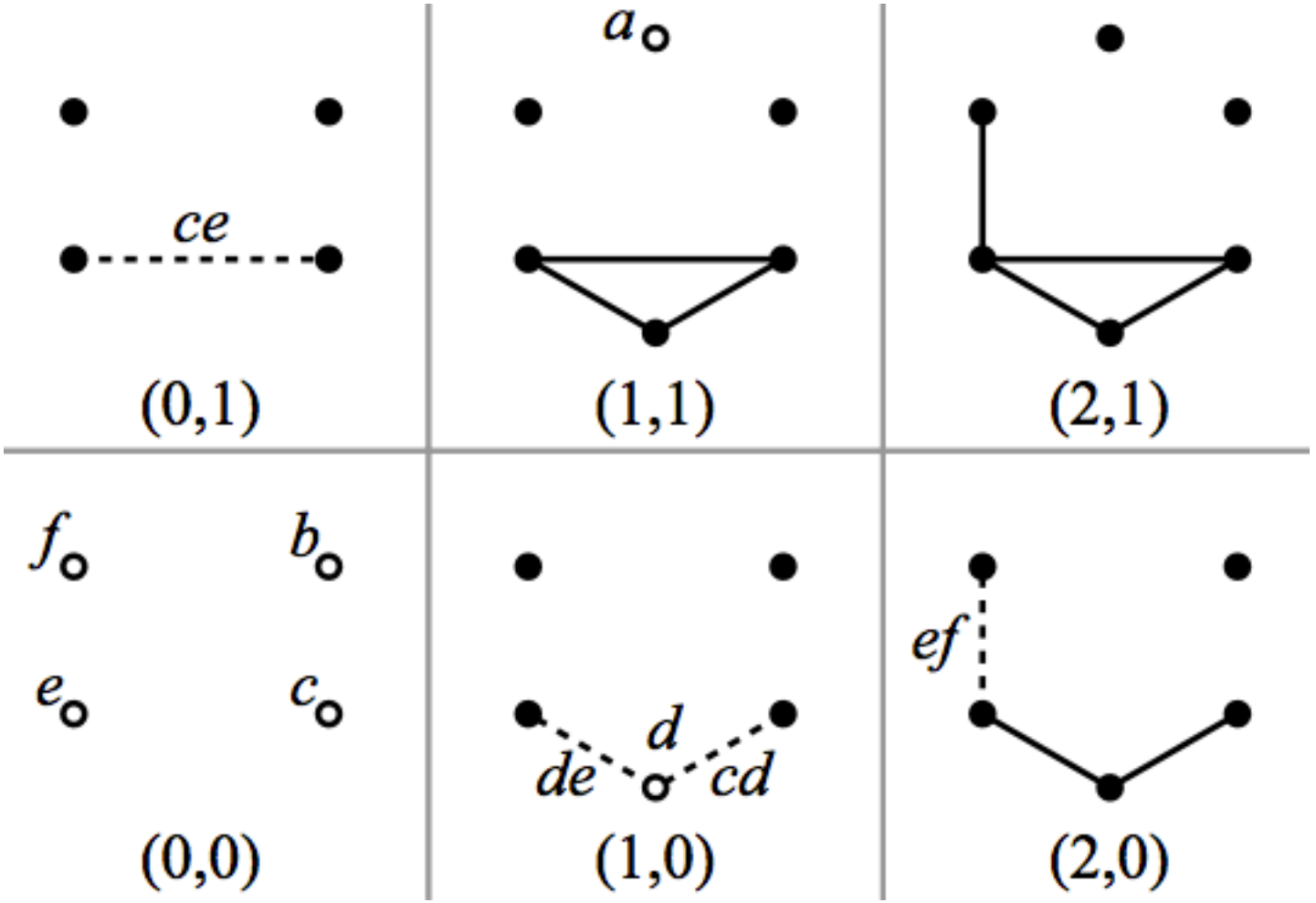


$Z_2[x_1, x_2]/\langle x_1, x_2 \rangle$

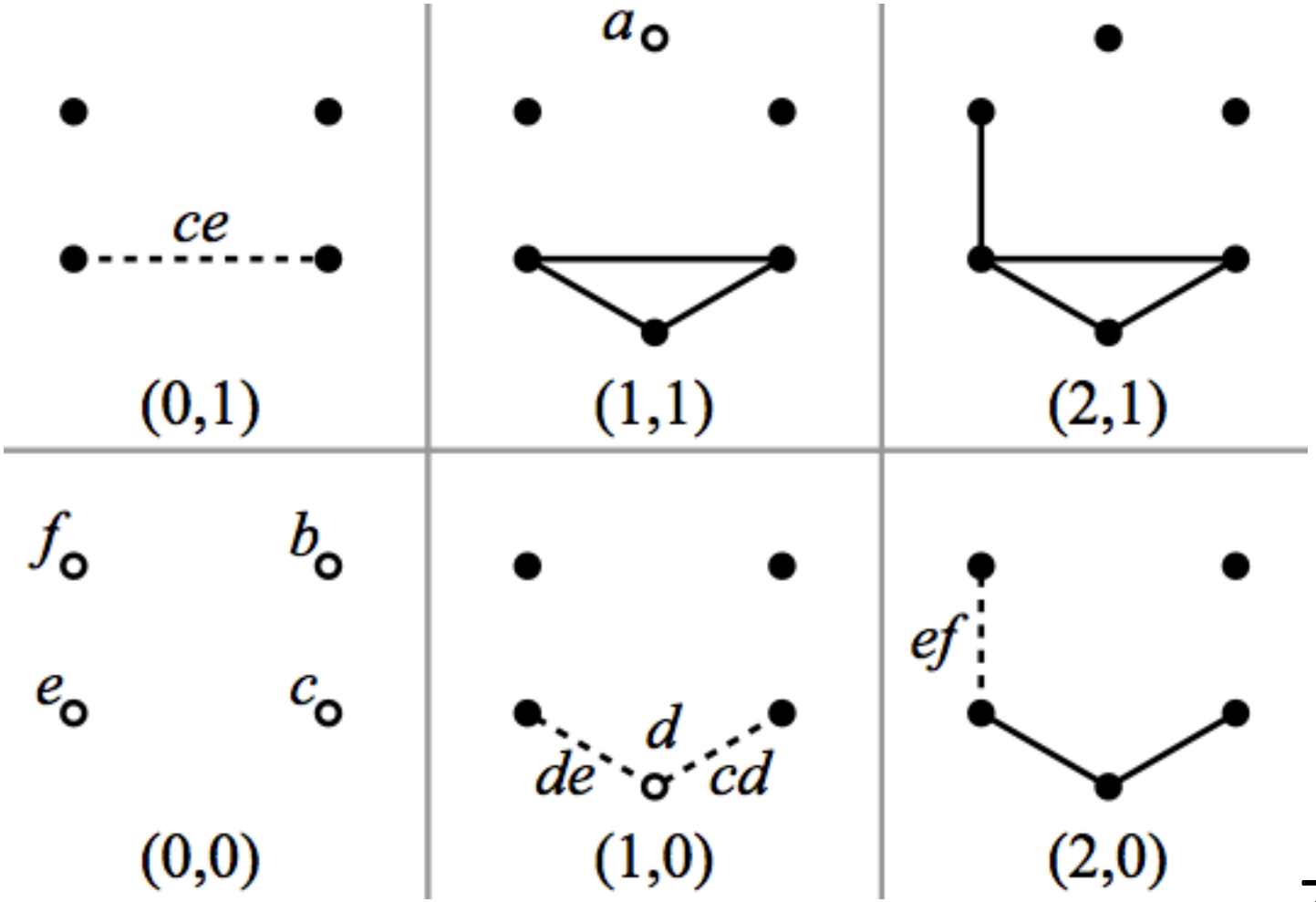
$$c^{(0,0)} \quad | \quad 0 \quad x_1 \quad 0 \quad x_1 \quad x_2 \quad 0$$

$$
\begin{array}{c}
 \\
a^{(1,1)} \\
d^{(1,0)} \\
b^{(0,0)} \\
c^{(0,0)} \\
e^{(0,0)} \\
f^{(0,0)}
\end{array}
\begin{array}{c}
\begin{array}{cccccc|cc}
ab & cd & bf & de+cd & ce & ef & bc+x_1^2bf+x_1x_2^2(de+cd)+x_2^2ef & af+ab+x_1bf \\
\hline
x_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
x_1x_2^2 & 0 & x_2^2 & 0 & 0 & 0 & 0 & 0 \\
0 & x_1 & 0 & x_1 & x_2 & 0 & 0 & 0 \\
0 & 0 & 0 & x_1 & x_2 & x_1^2 & 0 & 0 \\
0 & 0 & x_2^2 & 0 & 0 & x_1^2 & 0 & 0
\end{array}
\end{array}
$$





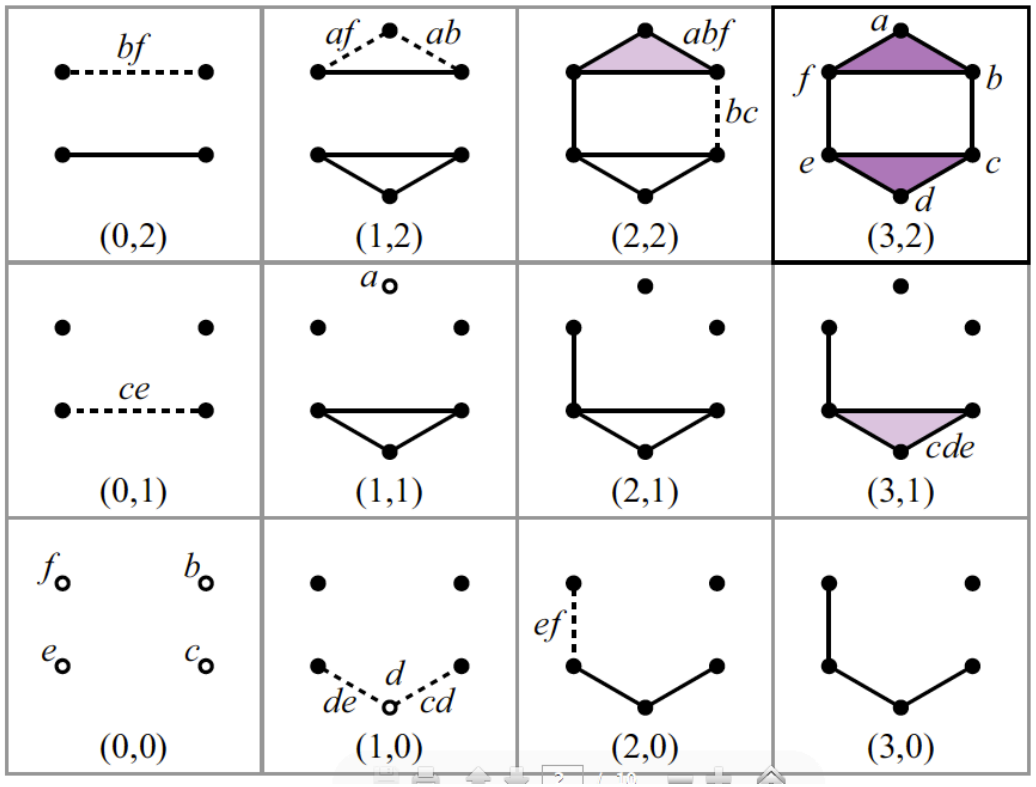
$$e^{(0,0)} \quad | \quad 0 \quad 0 \quad 0 \quad x_1 \quad x_2 \quad x_1^2$$



$Z_2[x_1, x_2]/x_1^2$

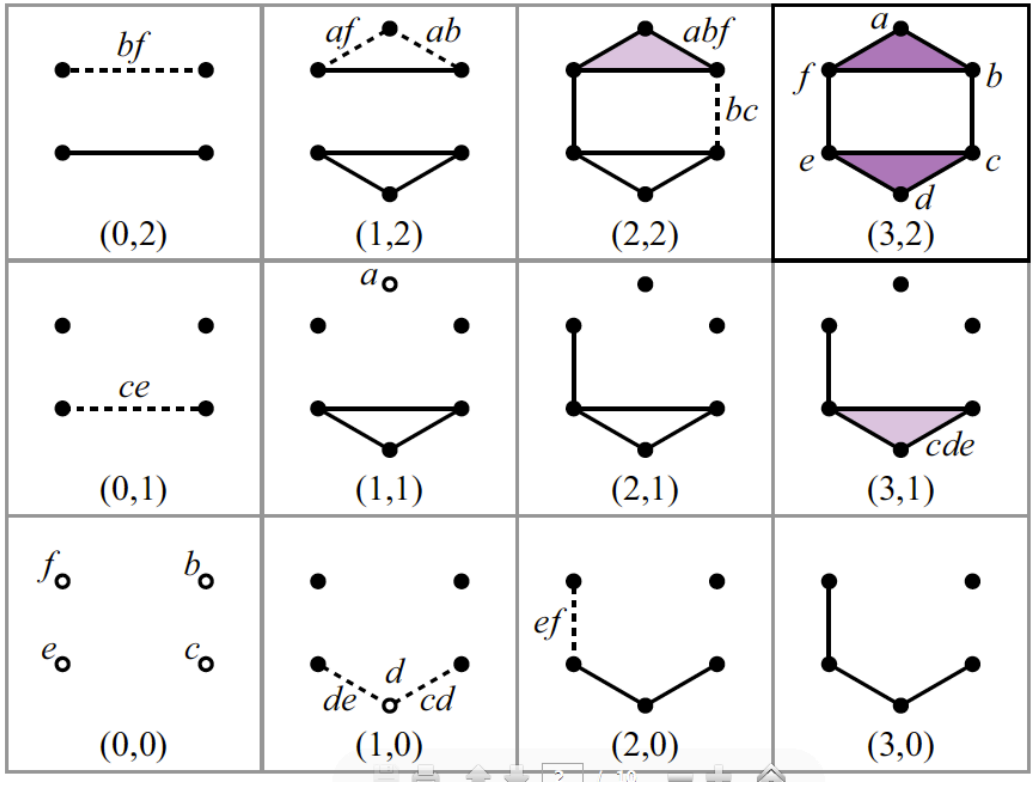
$$e^{(0,0)} \mid 0 \quad 0 \quad 0 \quad x_1 \quad x_2 \quad x_1^2$$

$$
\begin{array}{c}
 & ab & cd & bf & de+cd & ce & ef & bc+x_1^2bf+x_1x_2^2(de+cd)+x_2^2ef & af+ab+x_1bf \\
a^{(1,1)} & x_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
d^{(1,0)} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
b^{(0,0)} & x_1x_2^2 & 0 & x_2^2 & 0 & 0 & 0 & 0 & 0 \\
c^{(0,0)} & 0 & x_1 & 0 & x_1 & x_2 & 0 & 0 & 0 \\
e^{(0,0)} & 0 & 0 & 0 & x_1 & x_2 & x_1^2 & 0 & 0 \\
f^{(0,0)} & 0 & 0 & x_2^2 & 0 & 0 & x_1^2 & 0 & 0
\end{array}
$$



$$f^{(0,0)}$$

$$
\begin{array}{c}
 \\
a^{(1,1)} \\
d^{(1,0)} \\
b^{(0,0)} \\
c^{(0,0)} \\
e^{(0,0)} \\
f^{(0,0)}
\end{array}
\begin{array}{c}
\begin{array}{cccccc}
ab & cd & bf & de+cd & ce & ef
\end{array} \\
\begin{pmatrix}
x_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
x_1x_2^2 & 0 & x_2^2 & 0 & 0 & 0 \\
0 & x_1 & 0 & x_1 & x_2 & 0 \\
0 & 0 & 0 & x_1 & x_2 & x_1^2 \\
0 & 0 & x_2^2 & 0 & 0 & x_1^2
\end{pmatrix}
\end{array}
\begin{array}{c}
bc+x_1^2bf+x_1x_2^2(de+cd)+x_2^2ef \\
\begin{pmatrix}
0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0
\end{pmatrix}
\end{array}
\begin{array}{c}
af+ab+x_1bf \\
\begin{pmatrix}
0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0
\end{pmatrix}
\end{array}
$$



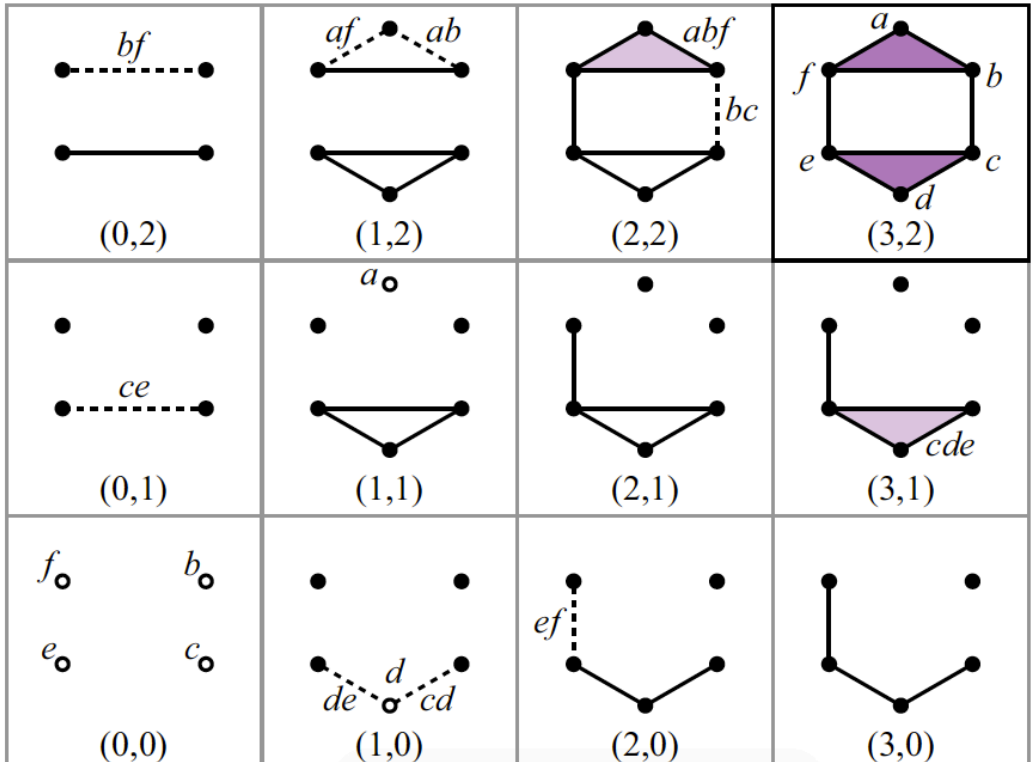
$$f^{(0,0)}$$

$$Z_2[x_1, x_2]$$

$$
\begin{array}{c}
 & ab & cd & bf & de+cd & ce & ef & bc+x_1^2bf+x_1x_2^2(de+cd)+x_2^2ef & af+ab+x_1bf \\
a^{(1,1)} & \left(x_2 \right. & 0 & 0 & 0 & 0 & 0 & 0 & \left. 0 \right) \\
d^{(1,0)} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
b^{(0,0)} & x_1x_2^2 & 0 & x_2^2 & 0 & 0 & 0 & 0 & 0 \\
c^{(0,0)} & 0 & x_1 & 0 & x_1 & x_2 & 0 & 0 & 0 \\
e^{(0,0)} & 0 & 0 & 0 & x_1 & x_2 & x_1^2 & 0 & 0 \\
f^{(0,0)} & 0 & 0 & x_2^2 & 0 & 0 & x_1^2 & 0 & \left. 0 \right)
\end{array}
$$

$$\Sigma^{(1,\,1)} Z_2[x_1, x_2]/x_2 \oplus Z_2[x_1, x_2]/x_2^2 \oplus Z_2[x_1, x_2]/x_2^2 \oplus$$

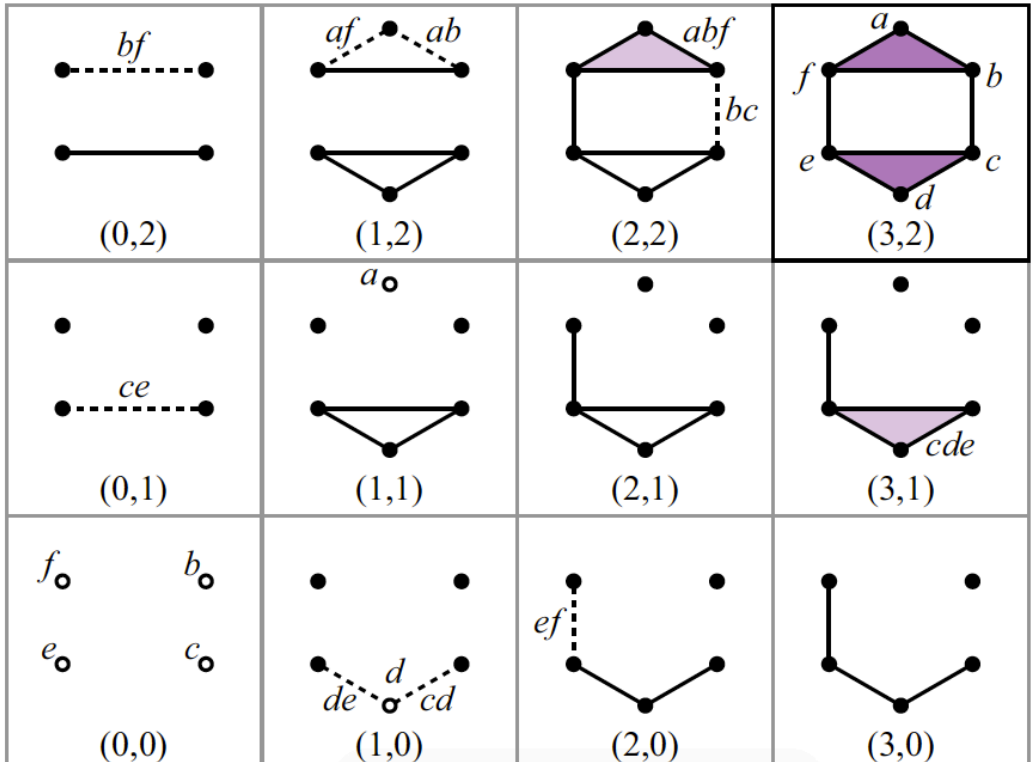
$$Z_2[x_1, x_2]/\langle x_1, x_2\rangle \oplus Z_2[x_1, x_2]/x_1^2 \oplus Z_2[x_1, x_2]$$



$$
\begin{array}{c}
 & ab & cd & bf & de+cd & ce & ef & bc+x_1^2bf+x_1x_2^2(de+cd)+x_2^2ef & af+ab+x_1bf \\
a^{(1,1)} & x_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
d^{(1,0)} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
b^{(0,0)} & x_1x_2^2 & 0 & x_2^2 & 0 & 0 & 0 & 0 & 0 \\
c^{(0,0)} & 0 & x_1 & 0 & x_1 & x_2 & 0 & 0 & 0 \\
e^{(0,0)} & 0 & 0 & 0 & x_1 & x_2 & x_1^2 & 0 & 0 \\
f^{(0,0)} & 0 & 0 & x_2^2 & 0 & 0 & x_1^2 & 0 & 0
\end{array}
$$

$$\Sigma^{(1,1)} Z_2[x_1, x_2]/x_2 \oplus Z_2[x_1, x_2]/x_2^2 \oplus Z_2[x_1, x_2]/x_2^2 \oplus$$

$$Z_2[x_1, x_2]/\langle x_1, x_2\rangle \oplus Z_2[x_1, x_2]/x_1^2 \oplus Z_2[x_1, x_2]$$



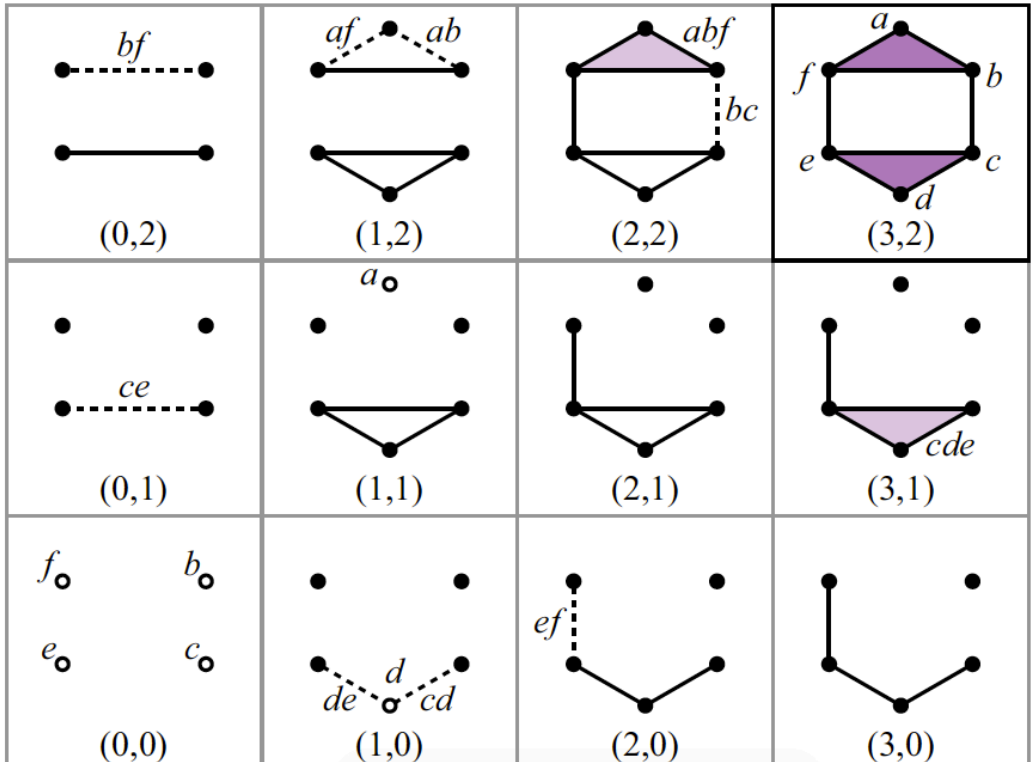
In general, use Buchberger's/ algorithm for constructing a Grobner basis.

Algorithm described in Computing Multidimensional Persistence, by Carlsson, Singh, Zomorodian, can be computed in polynomial time.

	ab	cd	bf	$de+cd$	ce	ef	$bc+x_1^2bf+x_1x_2^2(de+cd)+x_2^2ef$	$af+ab+x_1bf$
$a^{(1,1)}$	x_2	0	0	0	0	0	0	0
$d^{(1,0)}$	0	1	0	0	0	0	0	0
$b^{(0,0)}$	$x_1x_2^2$	0	x_2^2	0	0	0	0	0
$c^{(0,0)}$	0	x_1	0	x_1	x_2	0	0	0
$e^{(0,0)}$	0	0	0	x_1	x_2	x_1^2	0	0
$f^{(0,0)}$	0	0	x_2^2	0	0	x_1^2	0	0

$$\Sigma^{(1,1)} Z_2[x_1, x_2]/x_2 \oplus Z_2[x_1, x_2]/x_2^2 \oplus Z_2[x_1, x_2]/x_2^2 \oplus$$

$$Z_2[x_1, x_2]/\langle x_1, x_2\rangle \oplus Z_2[x_1, x_2]/x_1^2 \oplus Z_2[x_1, x_2]$$



In general, use Dionysus software,

http://www.mrzv.org/software/dionysus/

See next lecture,
1 week from today.