MATH:7450 (22M:305) Topics in Topology: Scientific and Engineering Applications of Algebraic Topology

Sept 23, 2013: Image data Application.

Fall 2013 course offered through the University of Iowa Division of Continuing Education

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http://www.math.uiowa.edu/~idarcy/AppliedTopology.html

Discriminative persistent homology of brain networks, 2011

Hyekyoung Lee Chung, M.K.; Hyejin Kang; Bung-Nyun Kim;Dong Soo Lee

Constructing functional brain networks with 97 regions of interest (ROIs)
extracted from FDG-PET data for
24 attention-deficit hyperactivity disorder (ADHD),
26 autism spectrum disorder (ASD) and
11 pediatric control (PedCon).

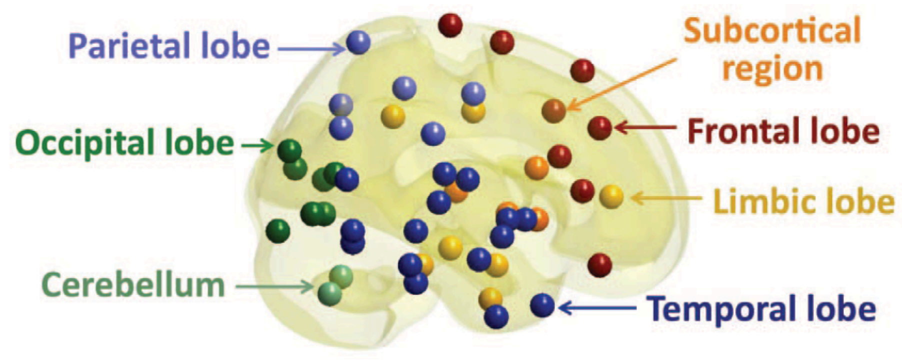


Fig. 3. Location of ROIs

Data = measurement f_j
taken at region j

Graph: 97 vertices representing 97 regions of interest
edge exists between two vertices i,j if correlation
between f_i and $f_j \geq$ threshold

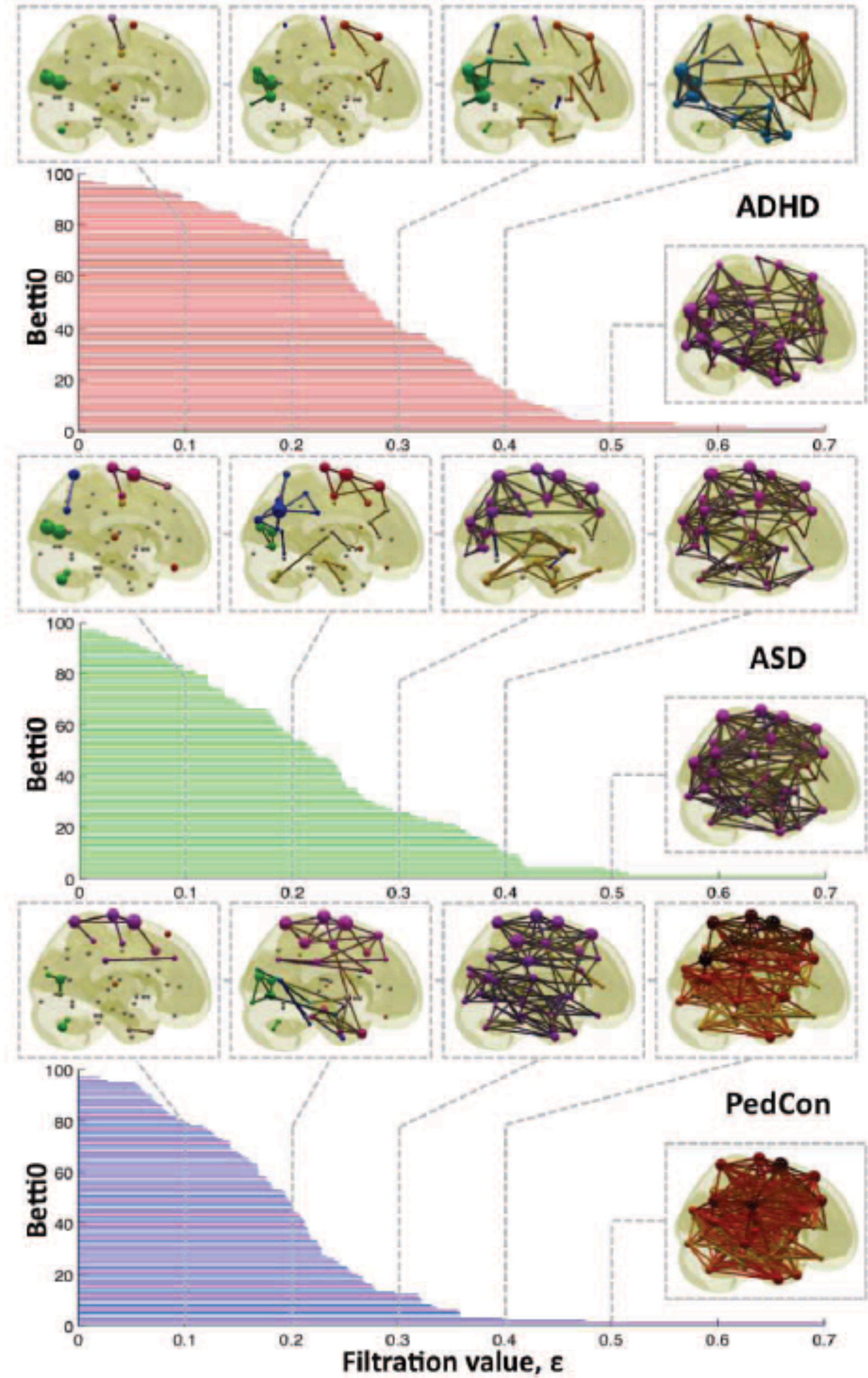
How to choose the threshold? Don't, instead use persistent homology

Vertices = Regions of Interest

Create Rips complex by growing epsilon balls (i.e. decreasing threshold) where distance between two vertices is given by

$$d(\boldsymbol{f}_i, \boldsymbol{f}_j) = 1 - \mathrm{corr}(\boldsymbol{f}_i, \boldsymbol{f}_j)$$

where $\boldsymbol{f}_i$ = measurement at location *i*



Lecture 9: Visualizing Data via Homology

Friday June 19, 2009

Coffee	
"Visualizing data via homology" image statistics data, range patches, neuroscience	**Gunnar Carlsson** (Stanford University)

http://www.ima.umn.edu/videos/?id=856

http://ima.umn.edu/2008-2009/ND6.15-26.09/activities/Carlsson-Gunnar/imafive-handout4up.pdf

Topological Methods for Large and Complex Data Sets

IMA Workshop on Machine Learning, Minneapolis

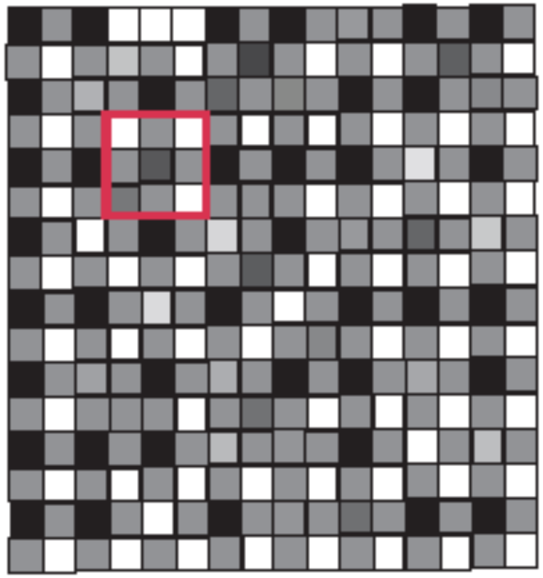
Application to Natural Image Statistics
With V. de Silva, T. Ishkanov, A. Zomorodian

Gunnar Carlsson, Stanford University

March 26, 2012

http://www.ima.umn.edu/videos/?id=1846

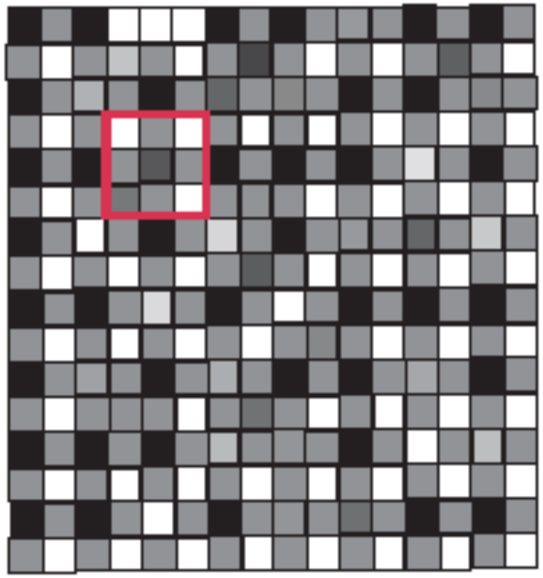
http://www.ima.umn.edu/2011-2012/W3.26-30.12/activities/Carlsson-Gunnar/imamachinefinal.pdf



An image taken by black and white digital camera can be viewed as a vector, with one coordinate for each pixel

Each pixel has a “gray scale” value, can be thought of as a real number (in reality, takes one of 255 values)

Typical camera uses tens of thousands of pixels, so images lie in a very high dimensional space, call it pixel space, P



Observations:

1. Each patch gives a vector in $\mathbb{R}^9$
2. Most patches will be nearly constant, or *low contrast*, because of the presence of regions of solid shading in most images

LOW CONTRAST HIGH CONTRAST

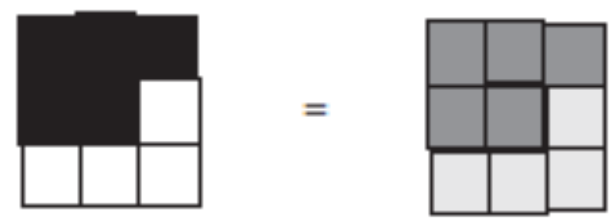
3. Low contrast will dominate statistics, not interesting

Lee-Mumford-Pedersen [LMP] study only high contrast patches.

Collection: 4.5 x 10^6 high contrast patches from a collection of images obtained by van Hateren and van der Schaaf

- Normalize mean intensity by subtracting mean from each pixel value to obtain patches with mean intensity $= 0$
- Puts data on an 8-dimensional hyperplane, $\cong \mathbb{R}^8$
- Means that we will consider as equivalent patches which can be obtained from each other by turning the intensity knob

- Normalize contrast by dividing by the D-norm, so obtain patches with D-norm $= 1$
- Means that data now lies on a 7-D ellipsoid, $\cong S^7$



Analysis

First Observation: The points fill out S^7 in the sense that every point in S^7 is "close" to a point in $\mathcal{M}$

Initially disappointing, since it means that nothing special can be said about the actual patches different from patches chosen at random

However, density of points varies a great deal from region to region

How to analyze?

Codensity

For integer $k > 0$, and PCD $\mathbb{X}$

$$\delta_k(x) = d(x, x')$$

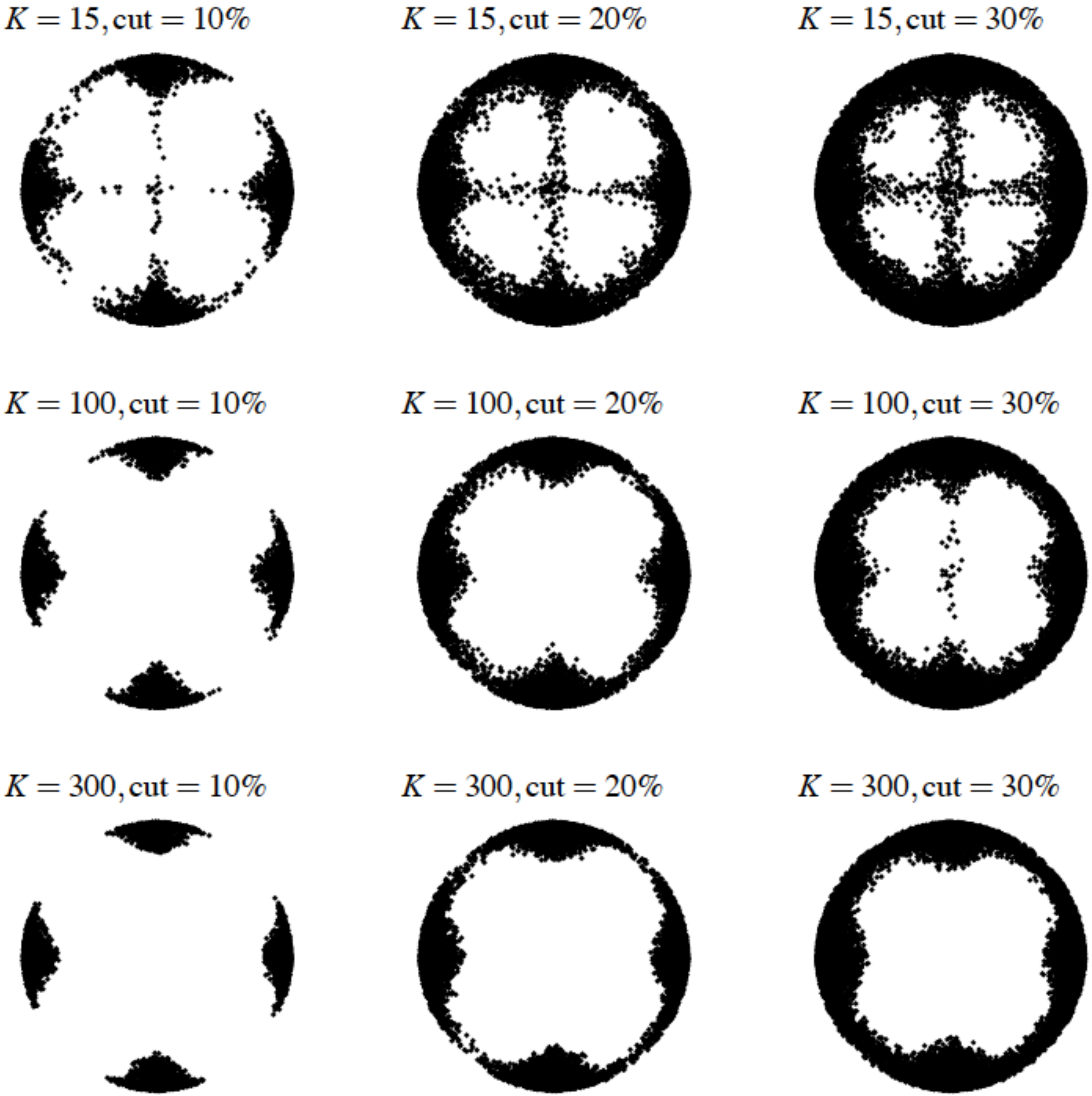
$$x' = \text{ any } k\text{-th nearest neighbor to } x \in \mathbb{X}$$

$$\delta_k(x) \text{ large} \implies x \text{ is sparse}$$

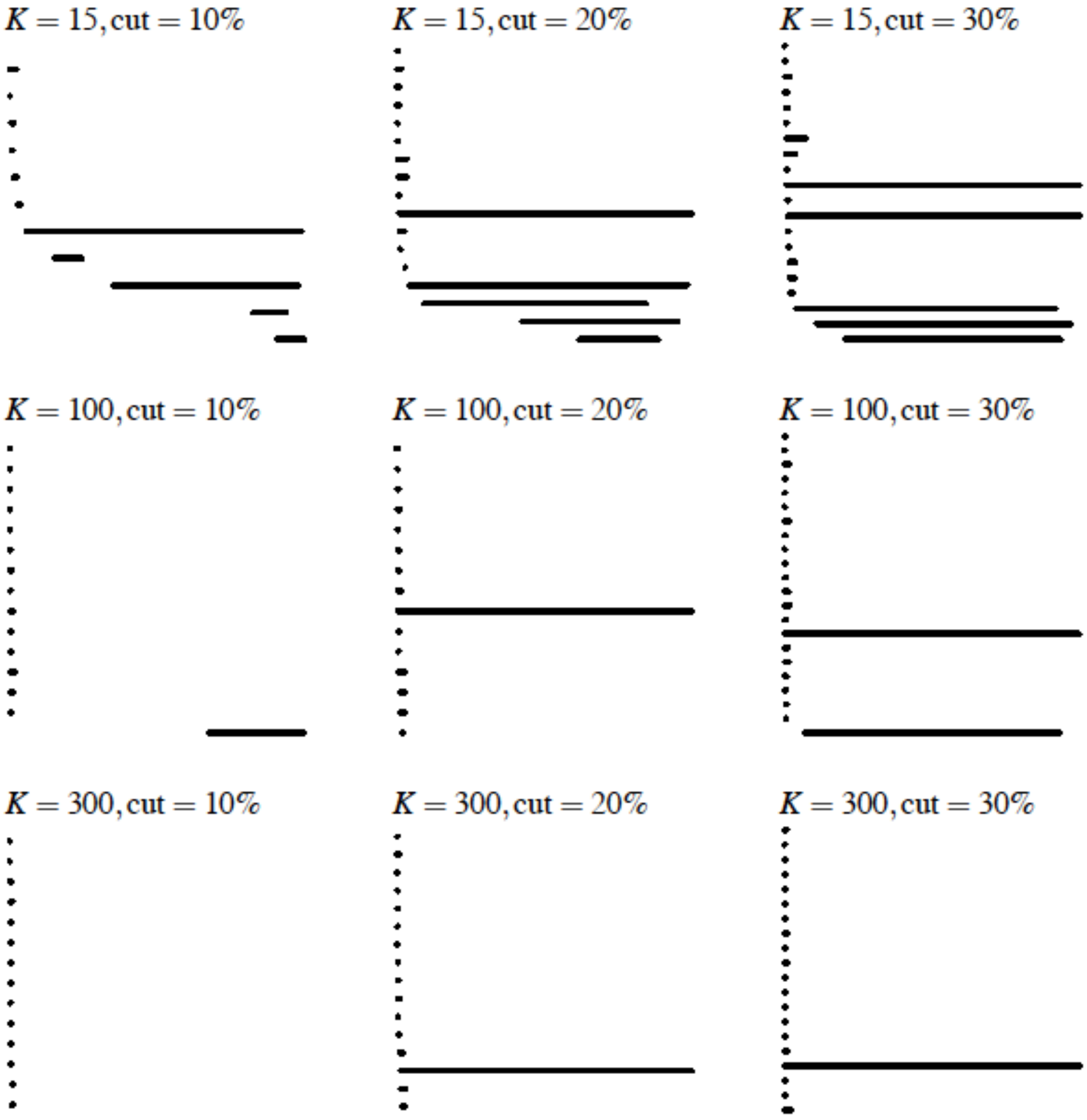
$$\delta_k(x) \text{ small} \implies x \text{ is dense}$$

$\mathcal{M}[k, T]$ is T% densest points as measured by δ_k

What is the persistent homology of these $\mathcal{M}[k, T]$?



Eurographics Symposium on Point-Based Graphics (2004)
Topological estimation using witness complexes
Vin de Silva and Gunnar Carlsson



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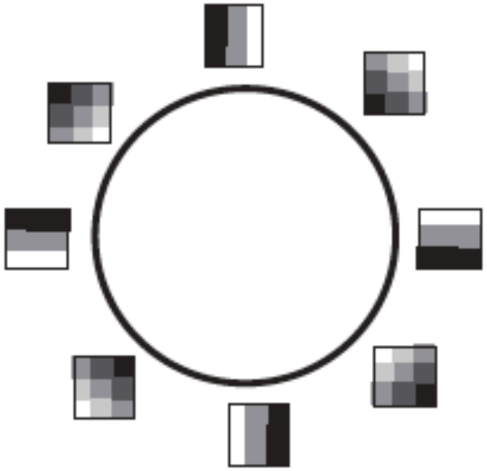
Primary Circle

5×10^4 points, $k = 300$, $T = 25$

One-dimensional barcode, suggests $\beta_1 = 1$

Is the set clustered around a circle?

Primary Circle



PRIMARY CIRCLE

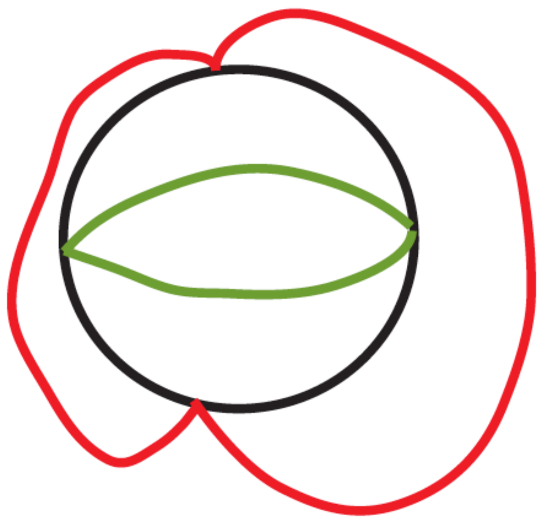
Three Circle Model

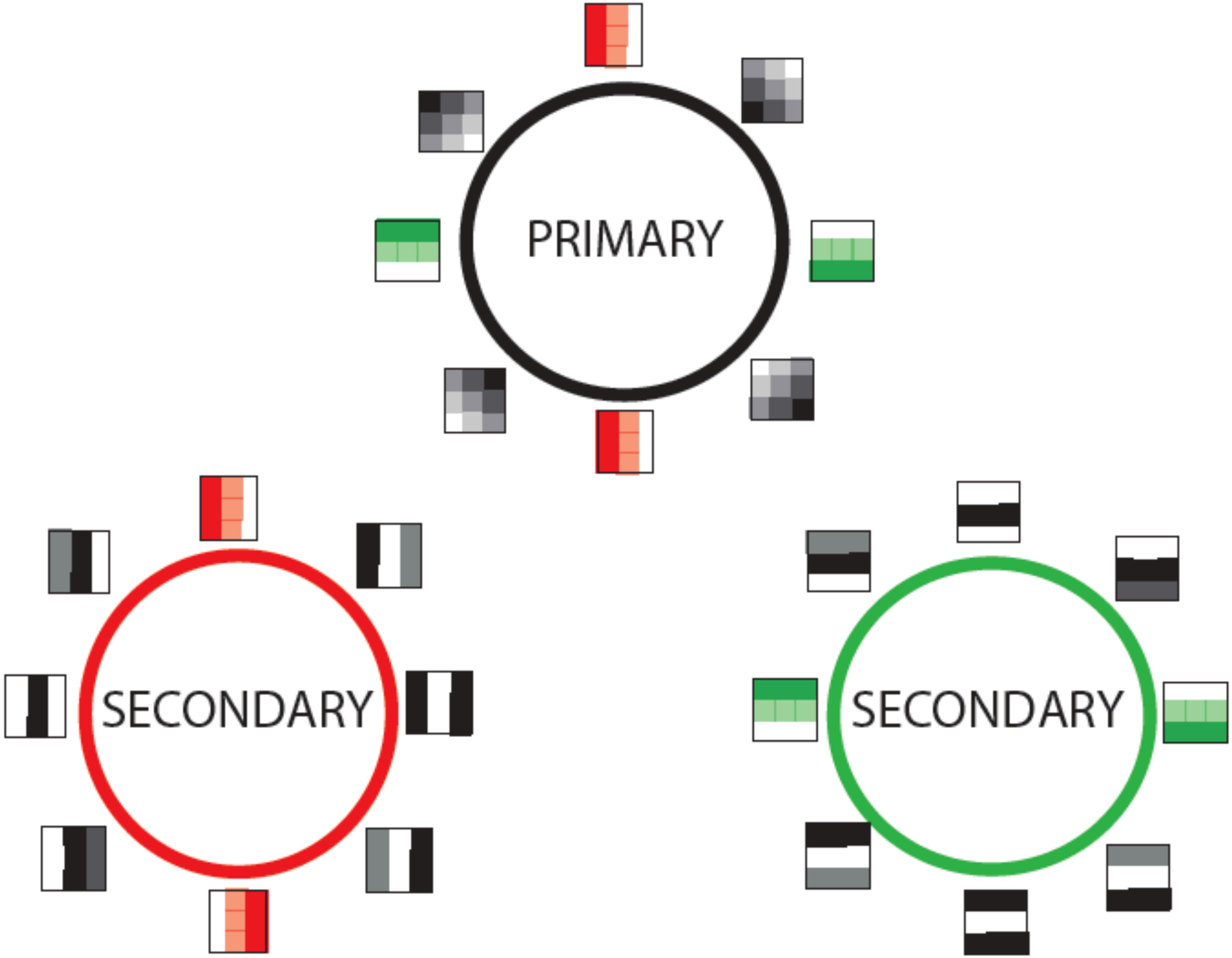
5×10^4 points, $k = 15$, $T = 25$

One-dimensional barcode, suggests $\beta_1 = 5$

What's the explanation for this?

Three Circle Model




PRIMARY
SECONDARY
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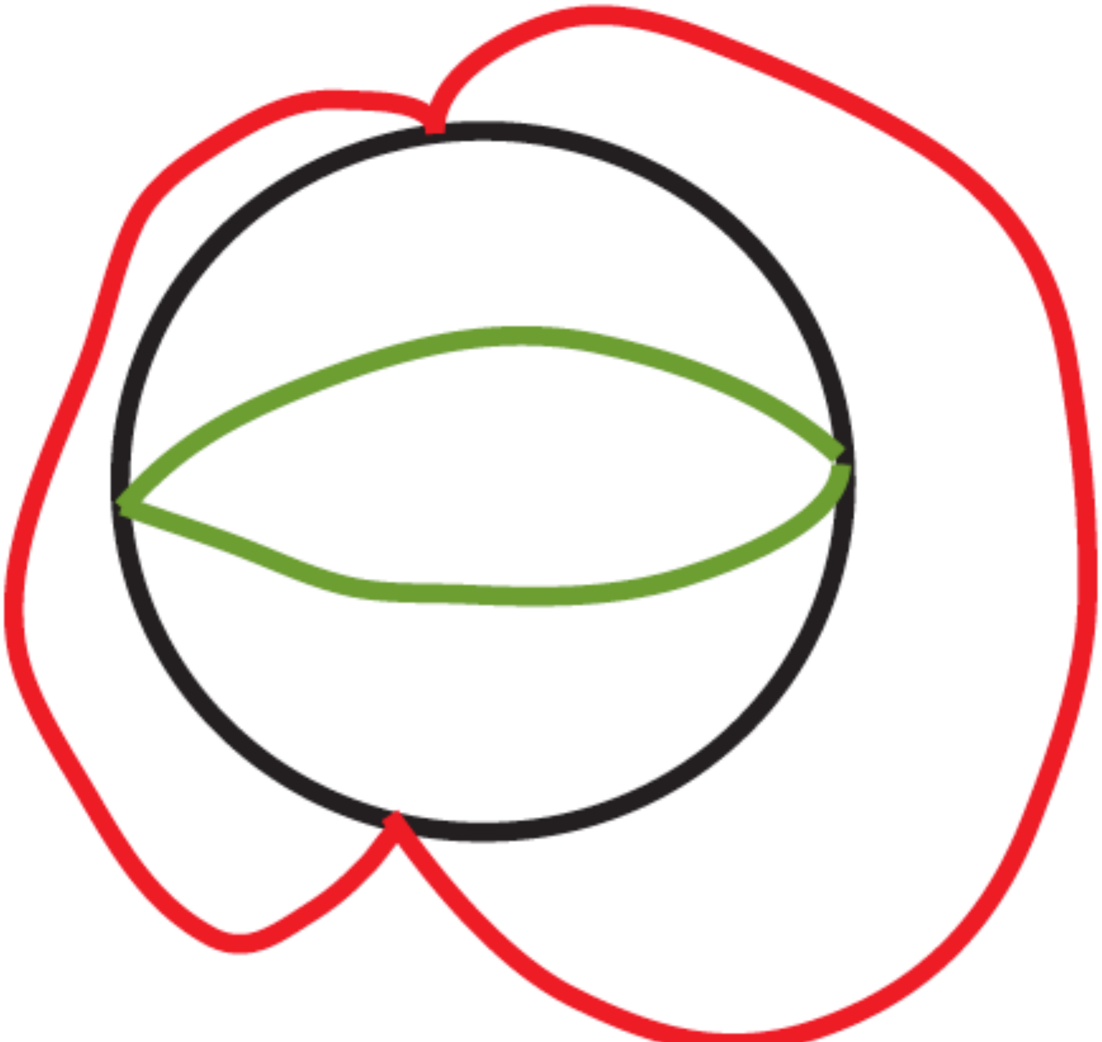
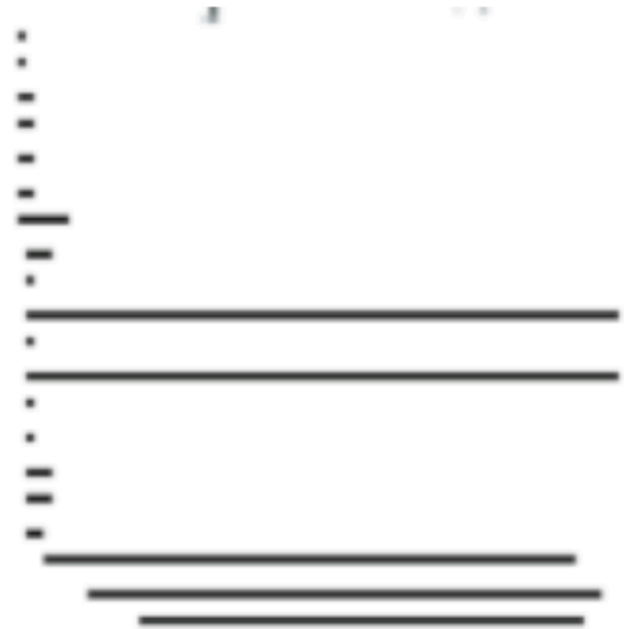
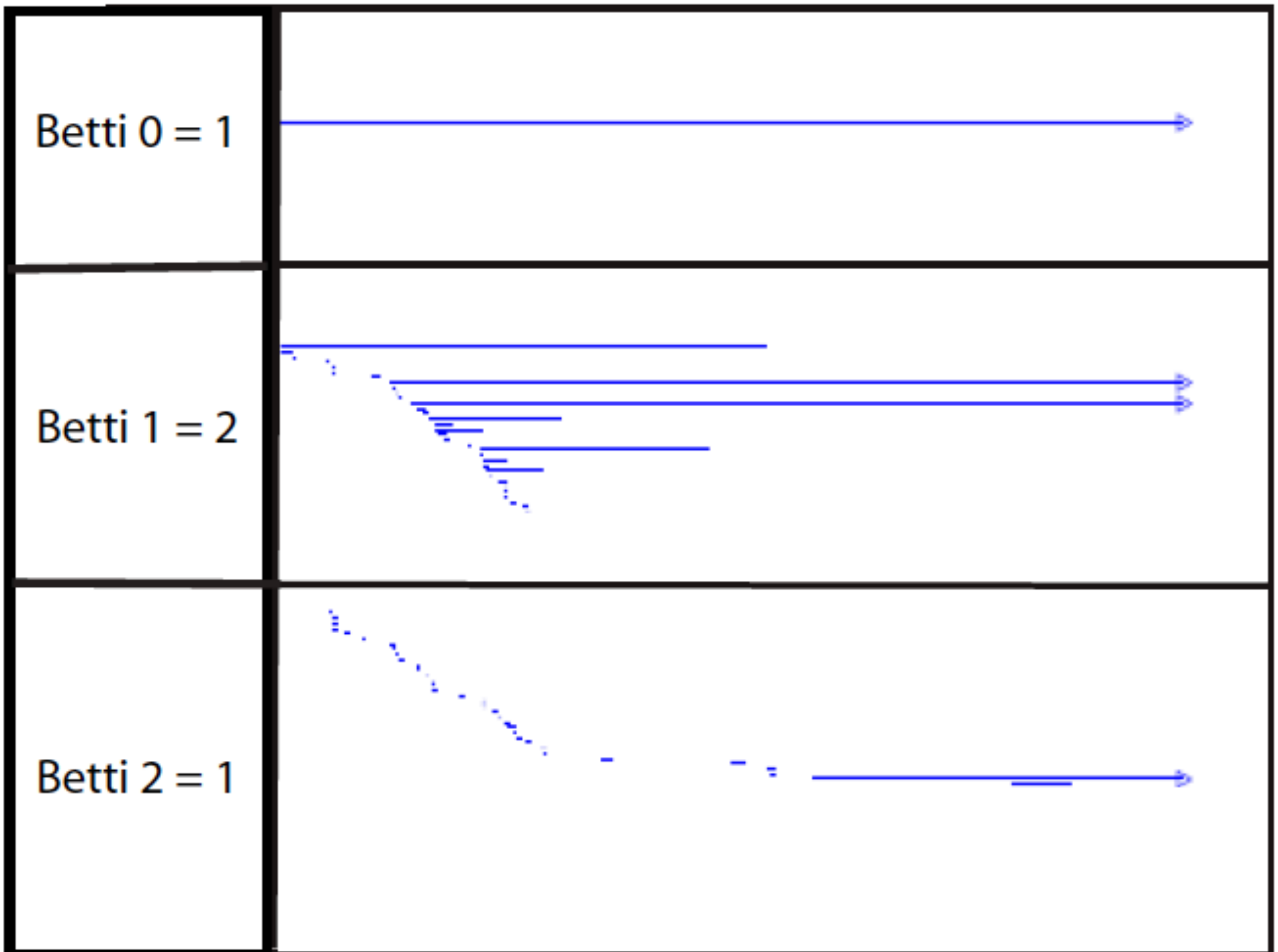
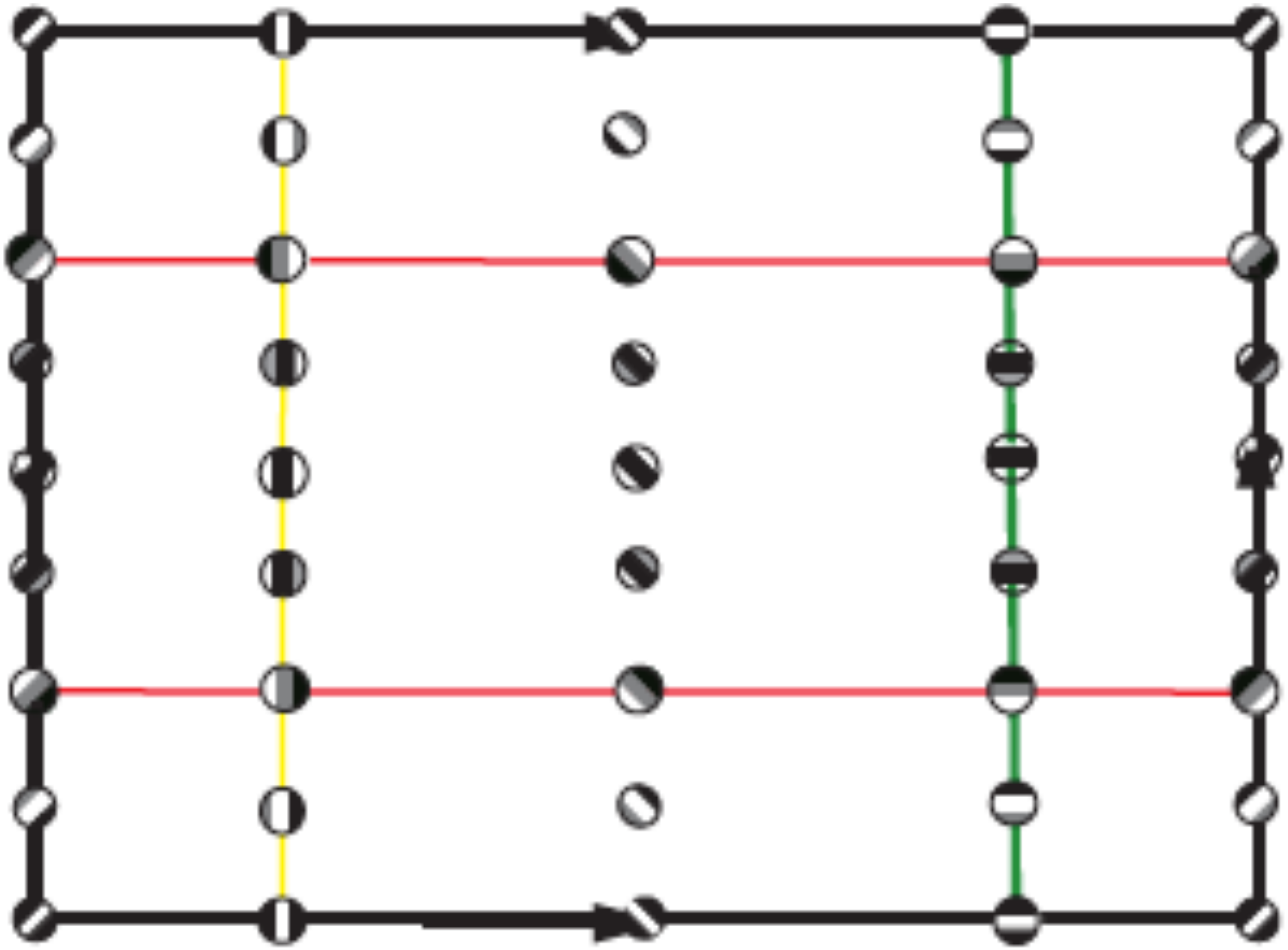


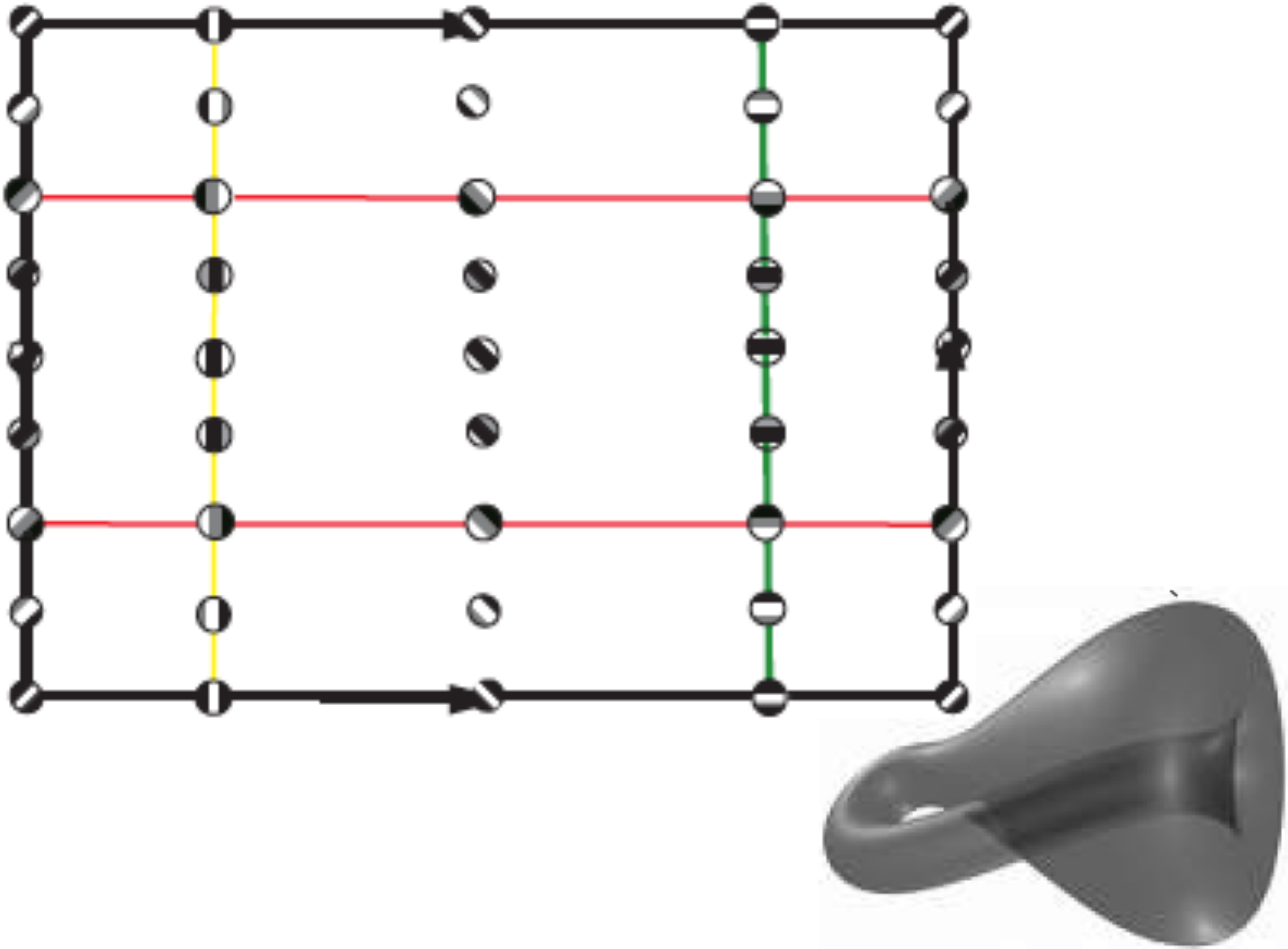


$\beta_1 = 5$

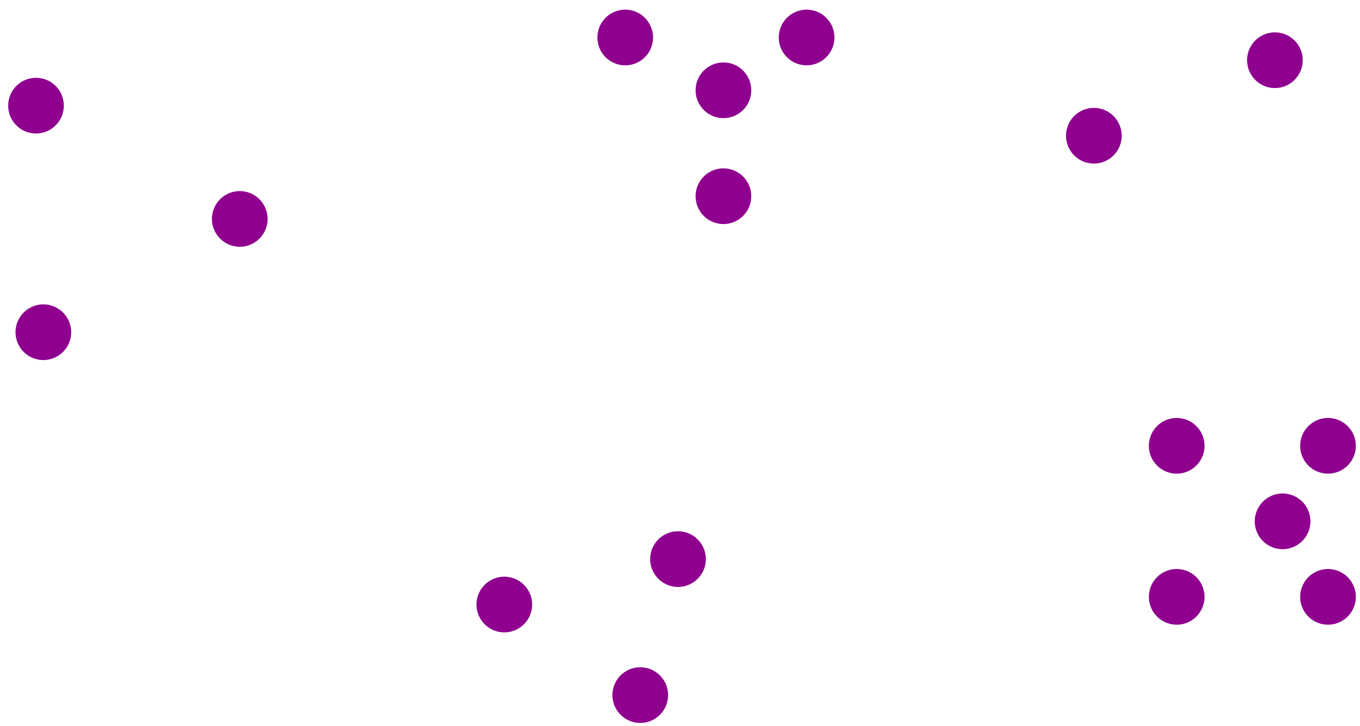
4.5×10^6 points, $k = 100$, $T = 10$







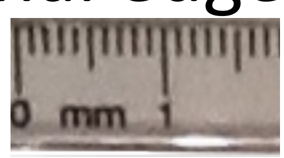
Creating a simplicial complex



Step 0.) Start by adding data points
= 0-dimensional vertices (0-simplices)

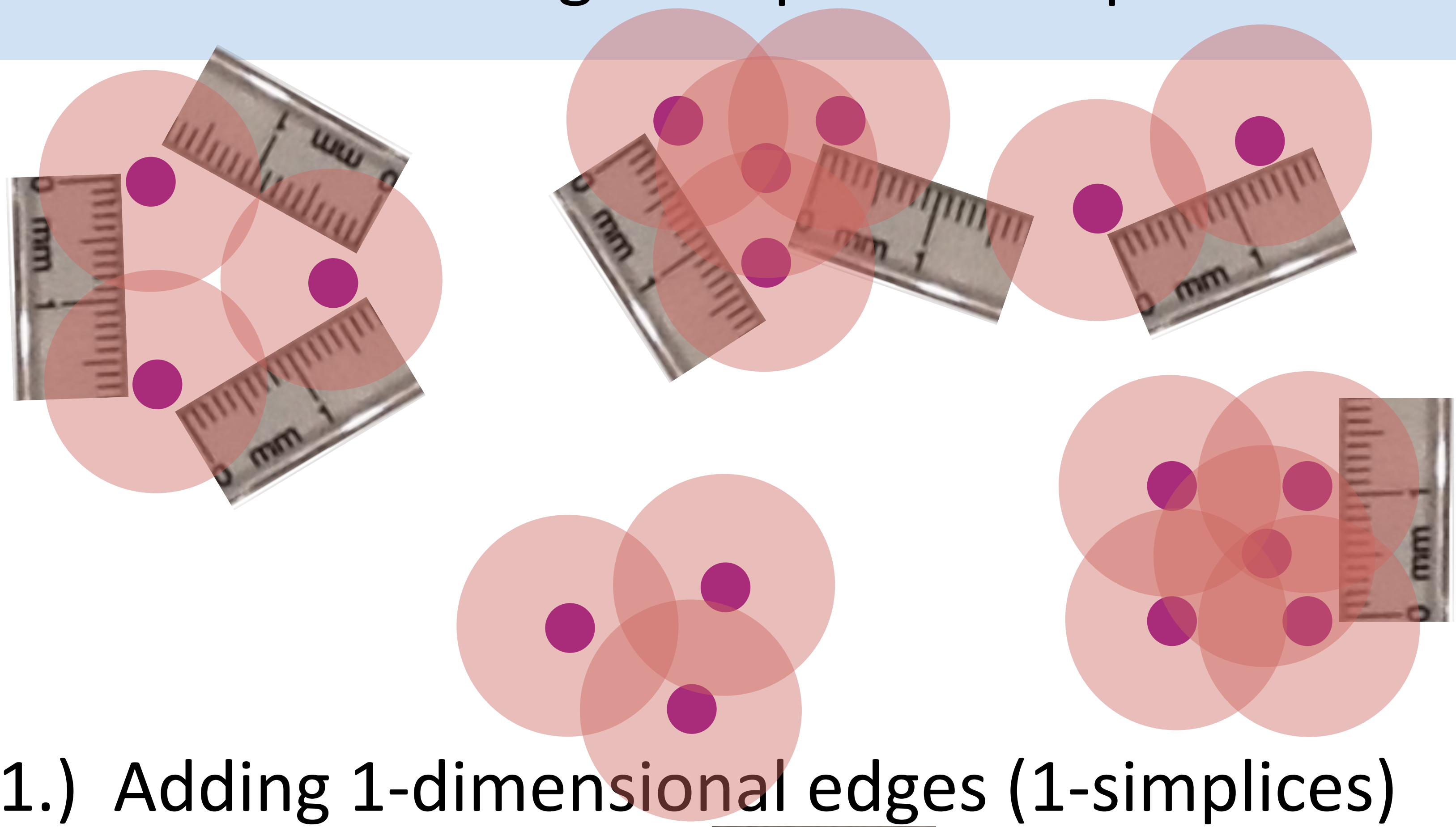
Creating a simplicial complex

1.) Adding 1-dimensional edges (1-simplices)

Let T = Threshold = 

Connect vertices v and w with an edge iff the distance between v and w is less than T

Creating a simplicial complex



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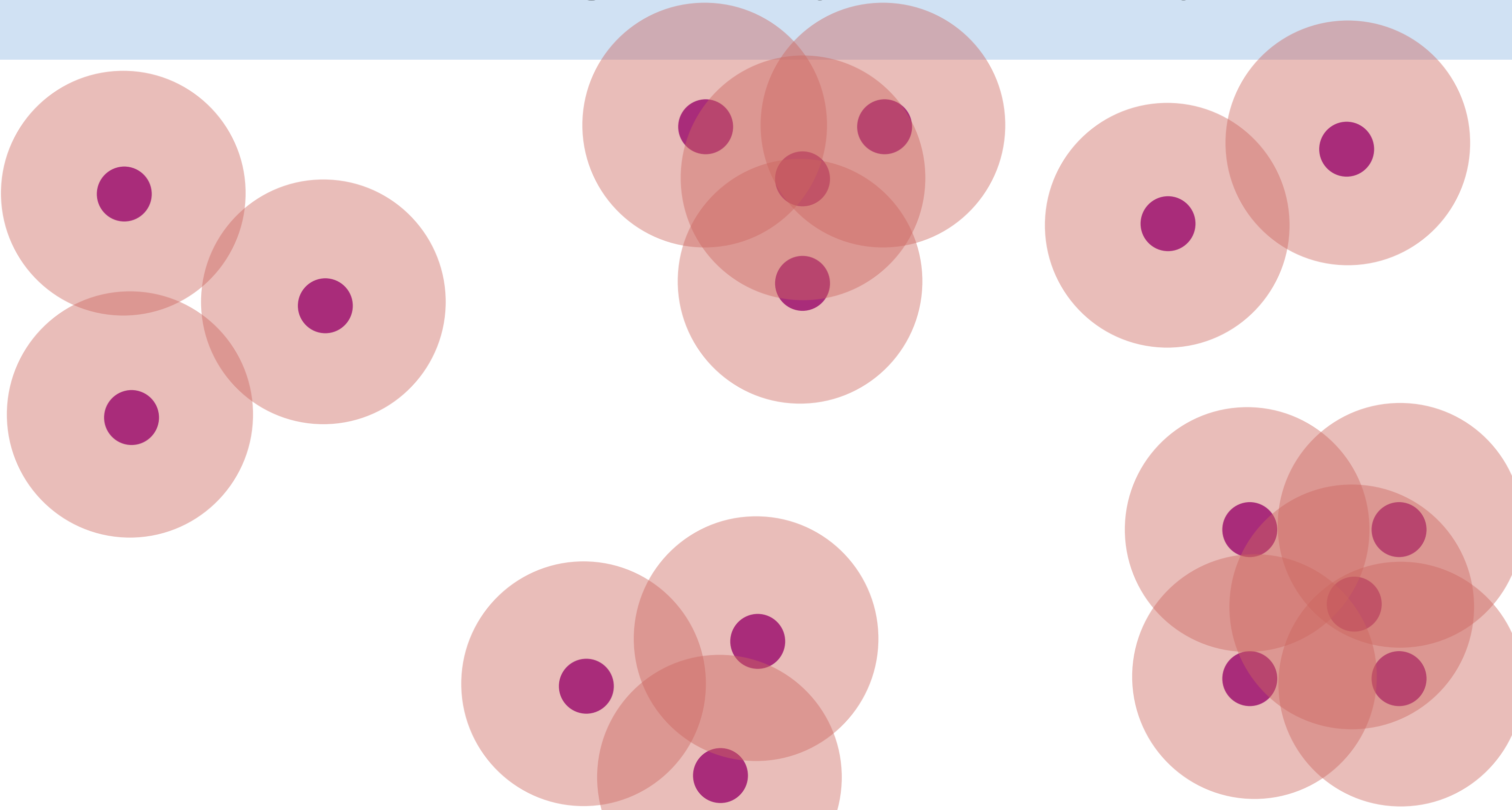

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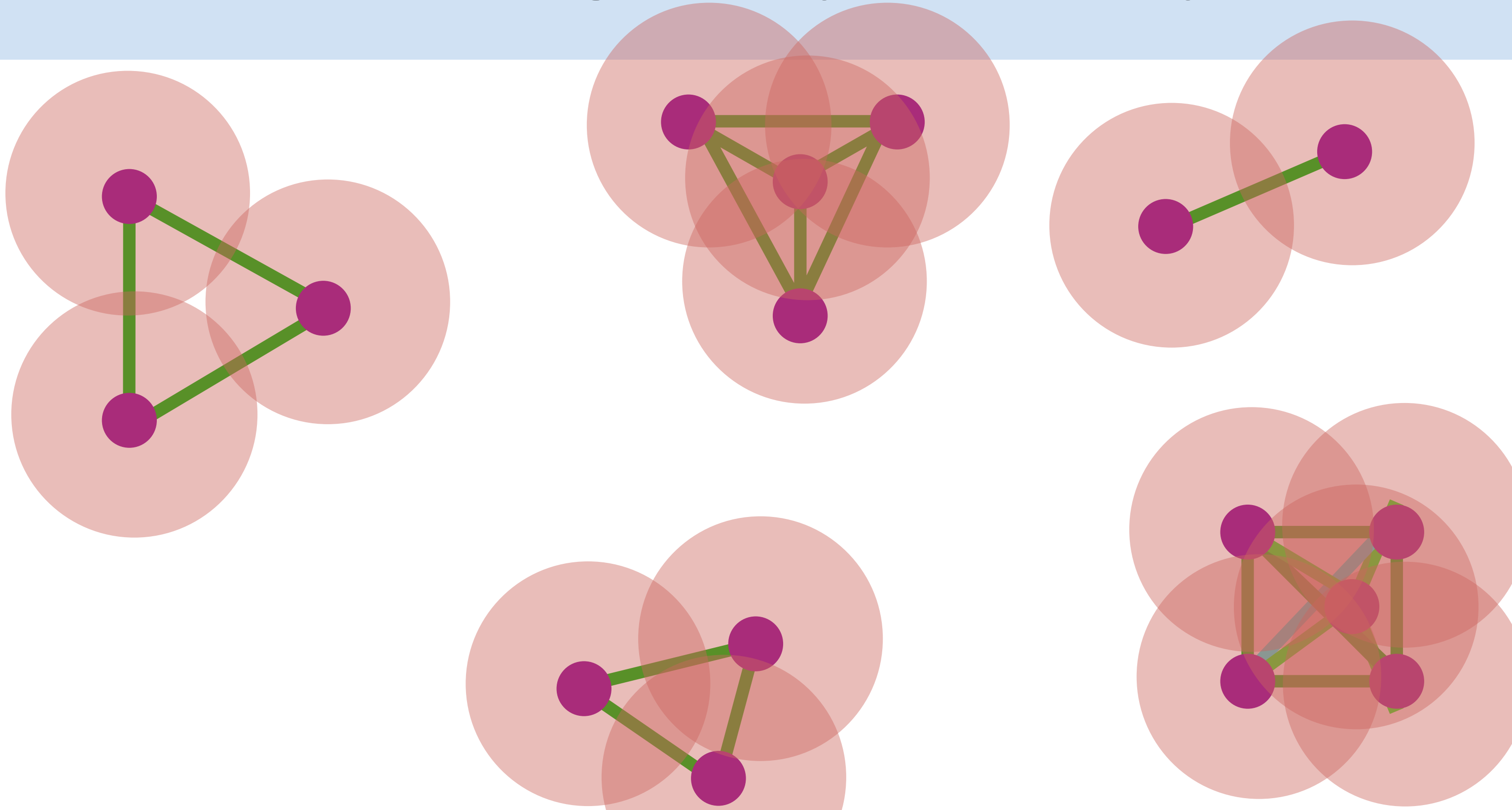
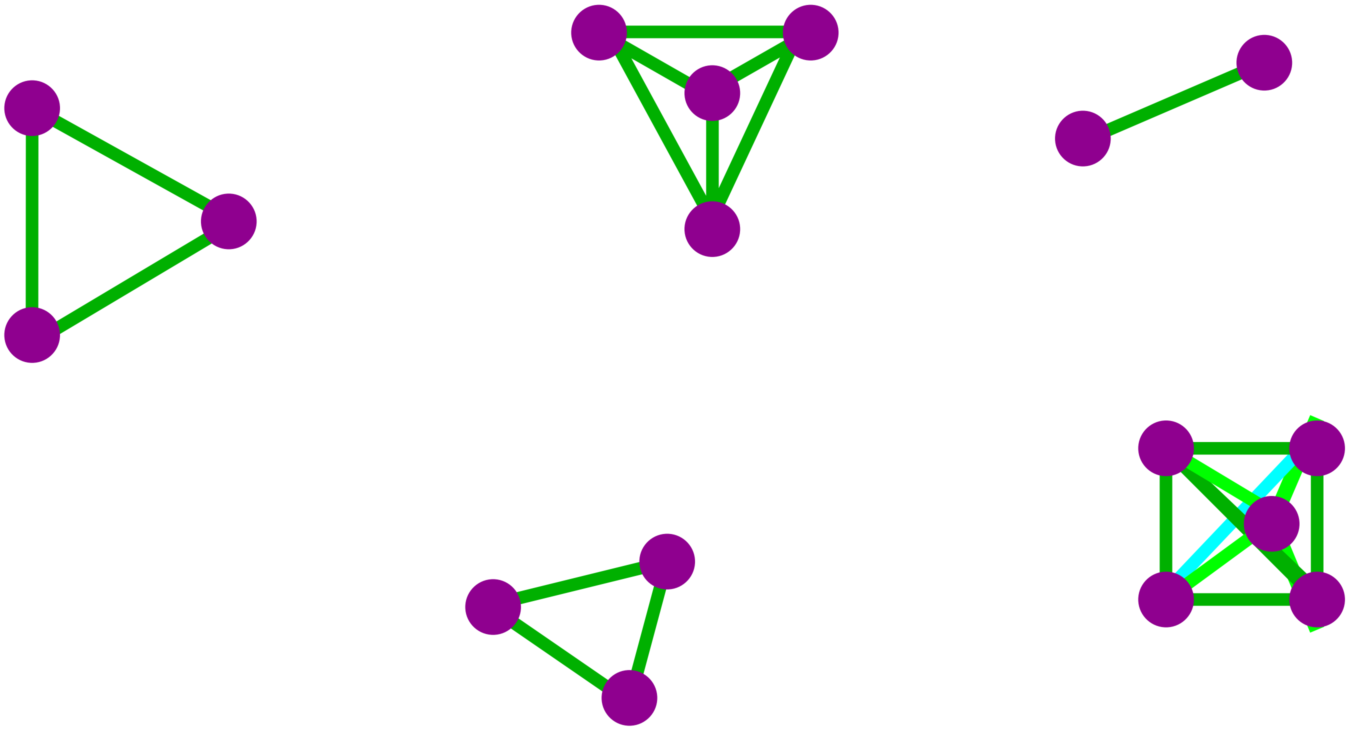


1.) Adding 1-dimensional edges (1-simplices)

Add an edge between data points that are “close”

Creating a simplicial complex



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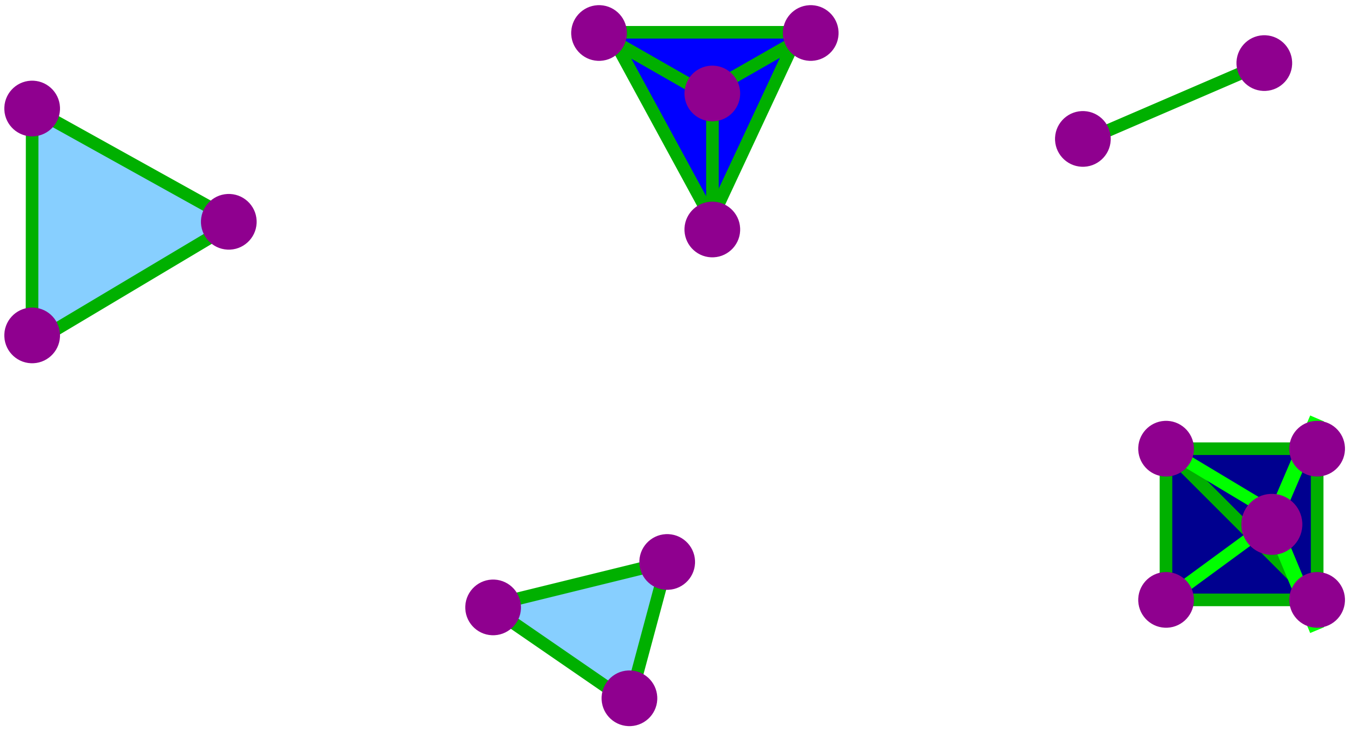
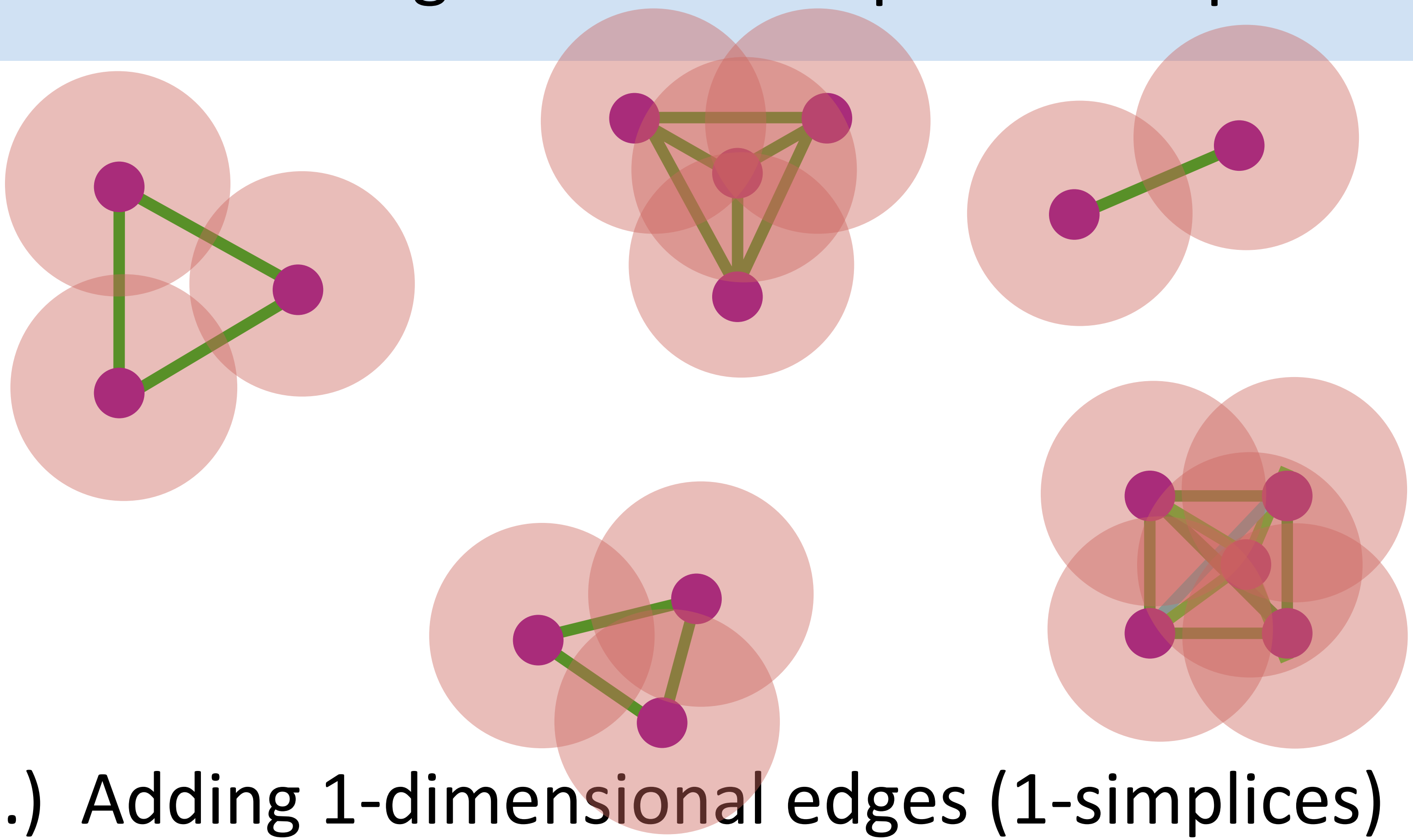
Add an edge between data points that are “close”

Vietoris Rips complex = flag complex = clique complex



2.) Add all possible simplices of dimensional > 1.

Creating the Čech simplicial complex



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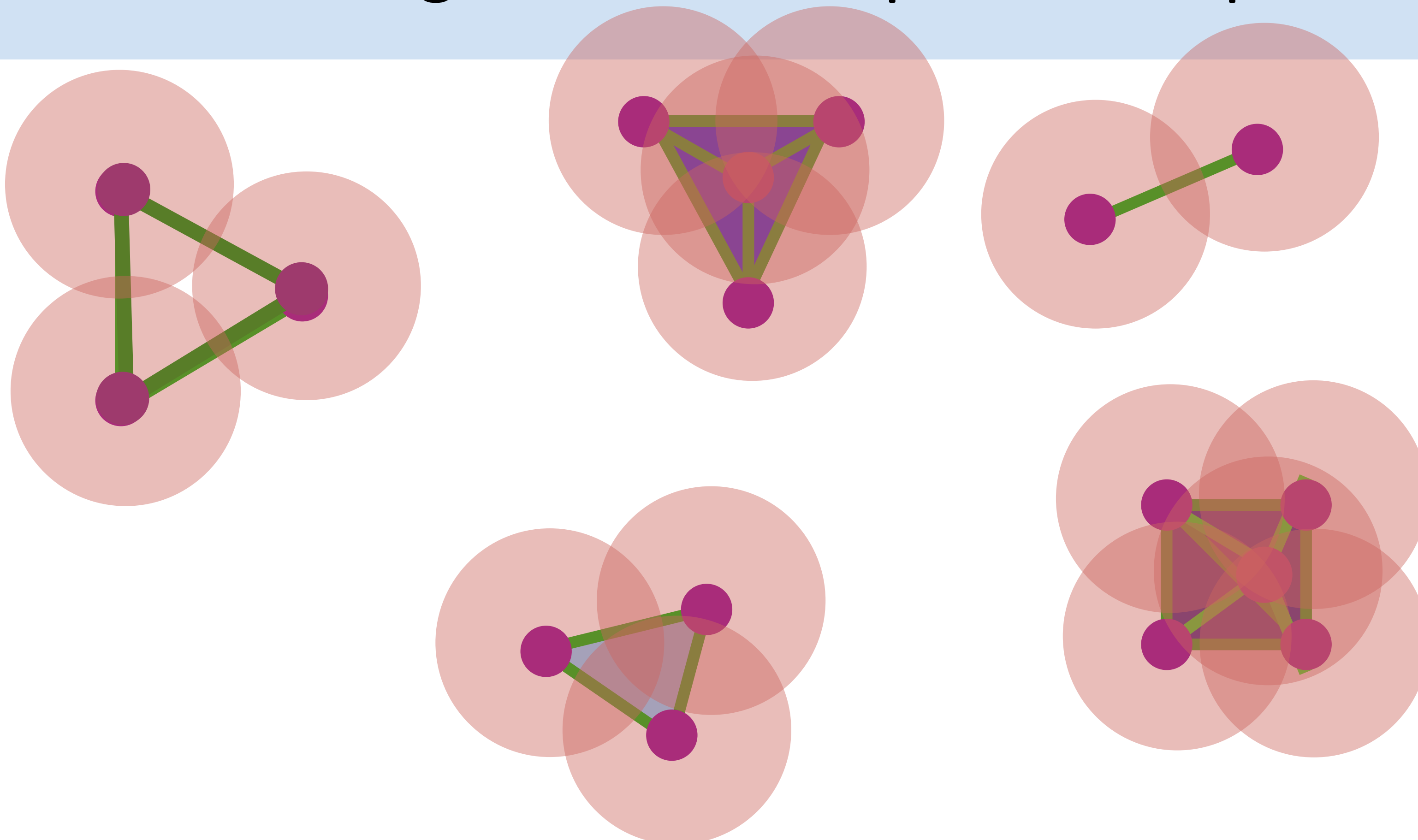
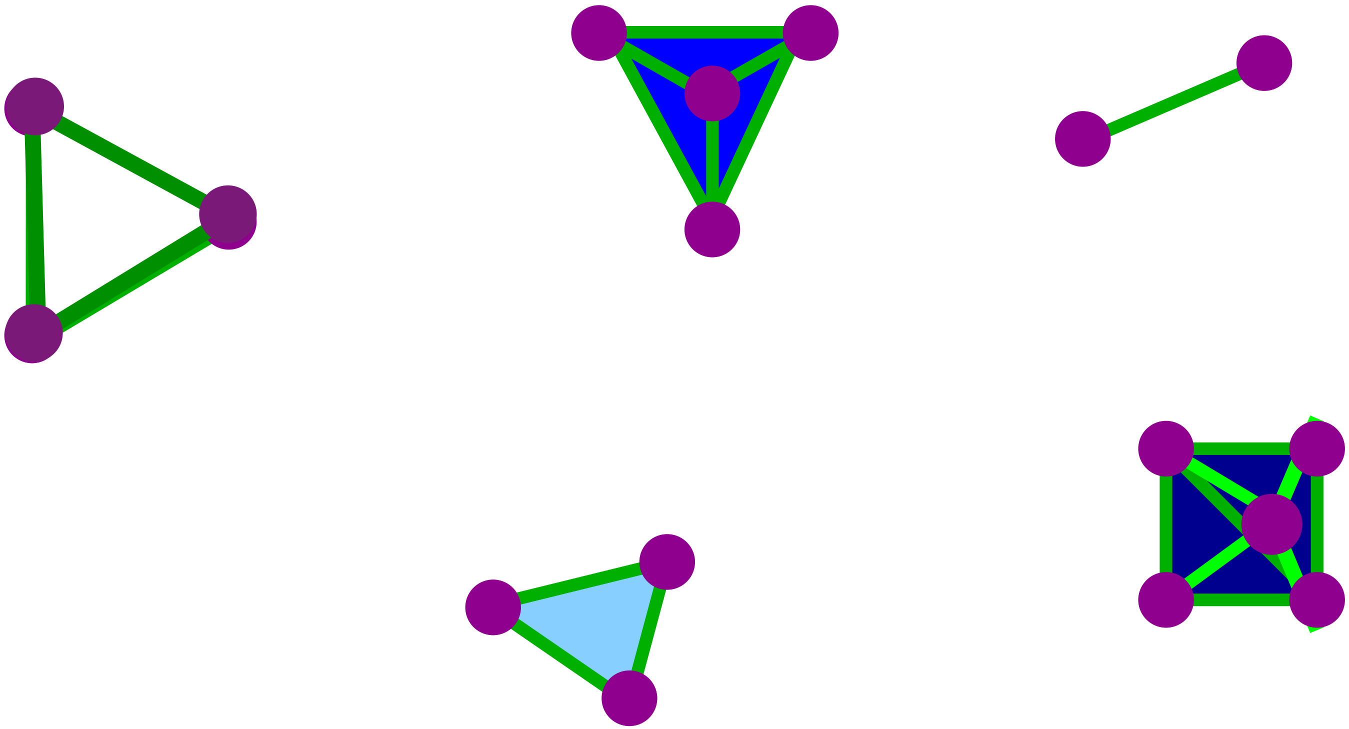
Creating the Čech simplicial complex



1.) $B_1 \cap \ldots \cap B_{k+1} \neq \emptyset$, create k-simplex $\{v_1, \ldots, v_{k+1}\}$.

Creating the Čech simplicial complex



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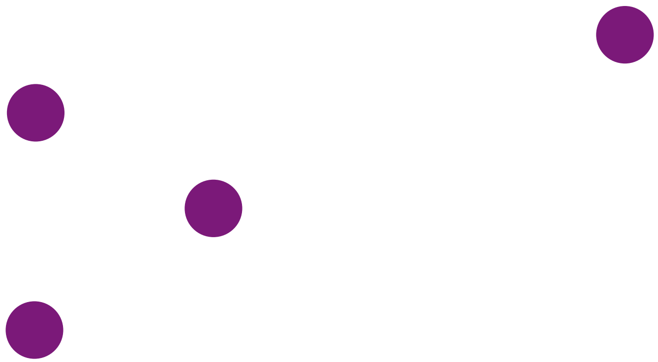
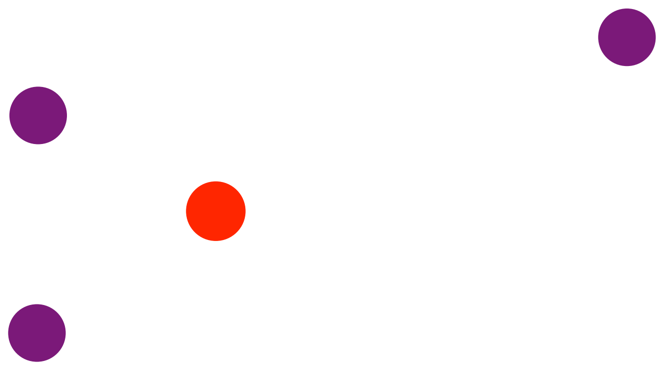


Creating Delaunay triangulation via Voronoi diagrams



Voronoi diagram:

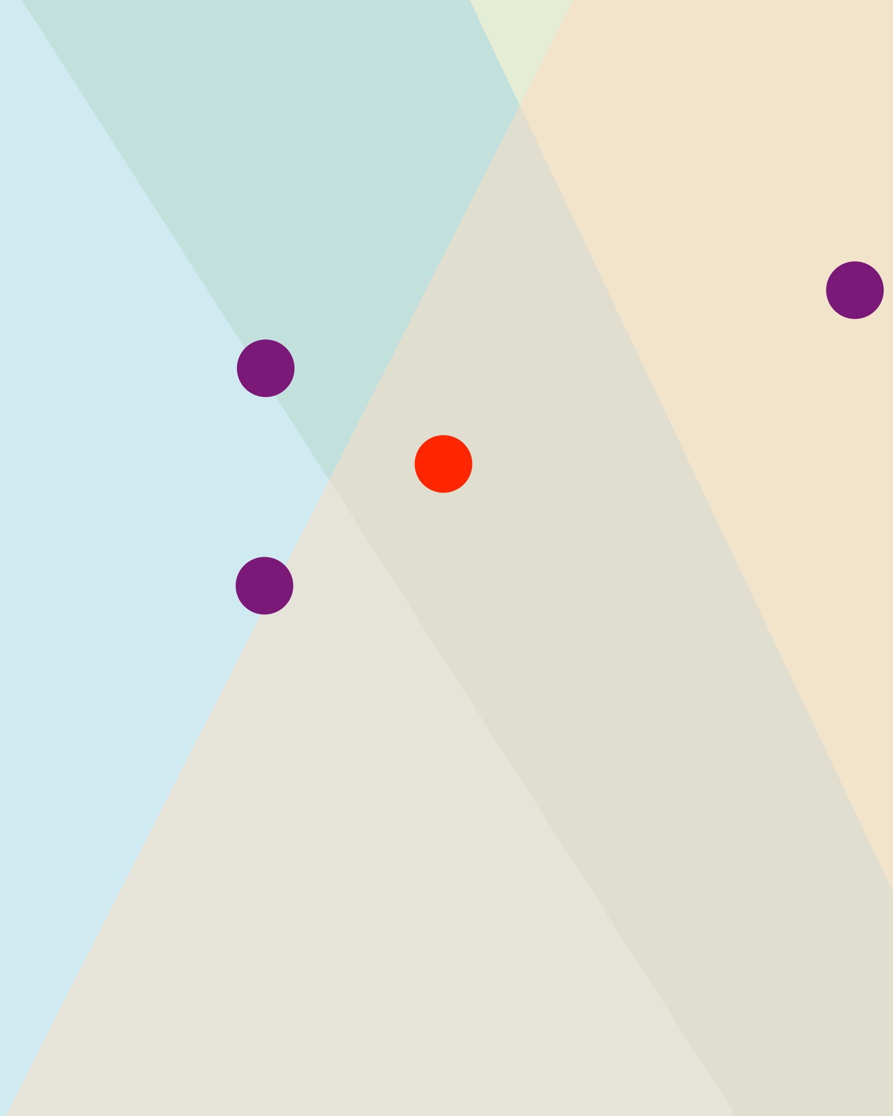
Suppose your data points live in R^n.

Choose data point v.

The *Voronoi cell* associated with v is

$$\bigcap_{w \neq v} H(v,w)$$

$$H(v,w) = \{ x \text{ in } R^n : d(x, v) \leq d(x, w) \}$$



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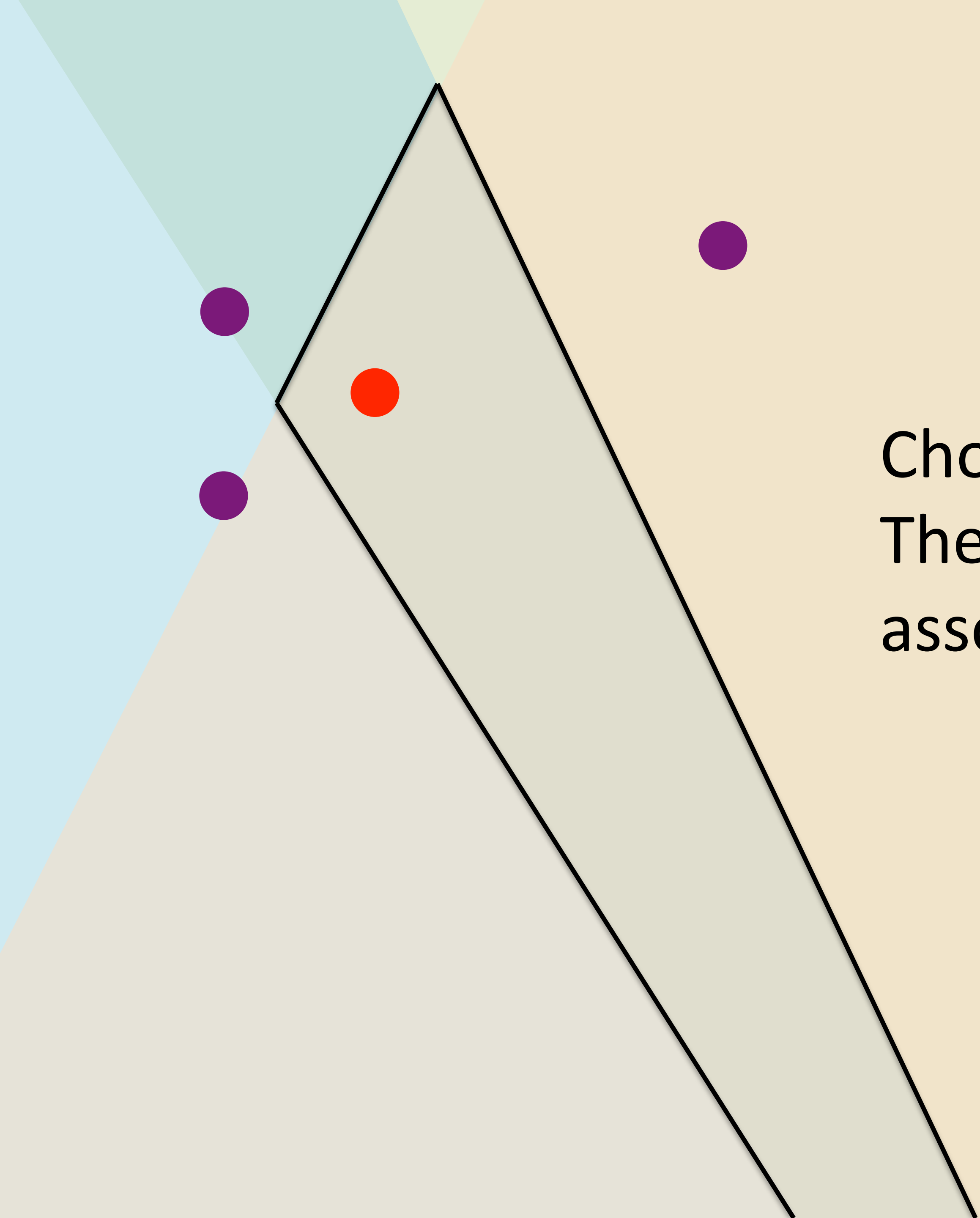

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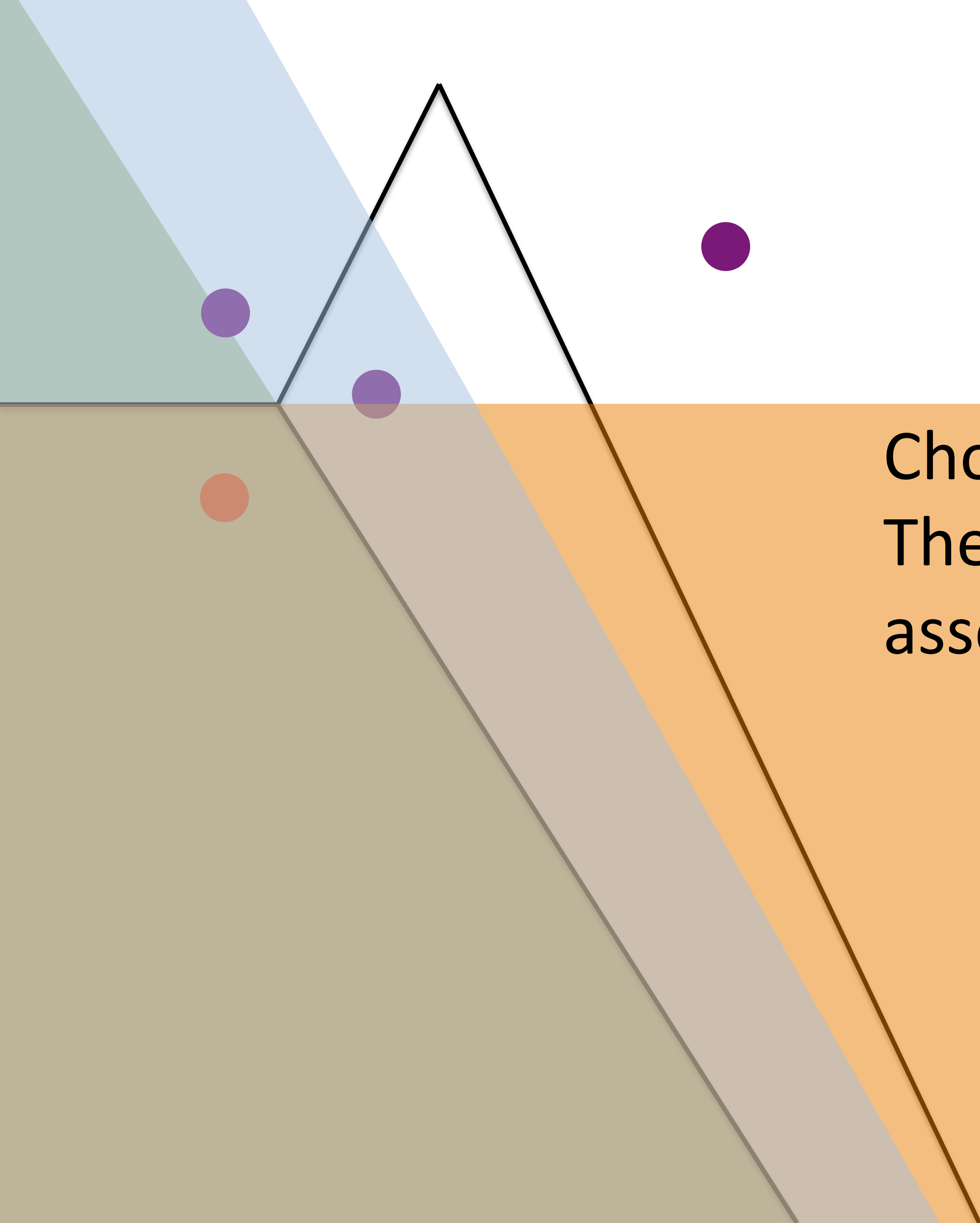

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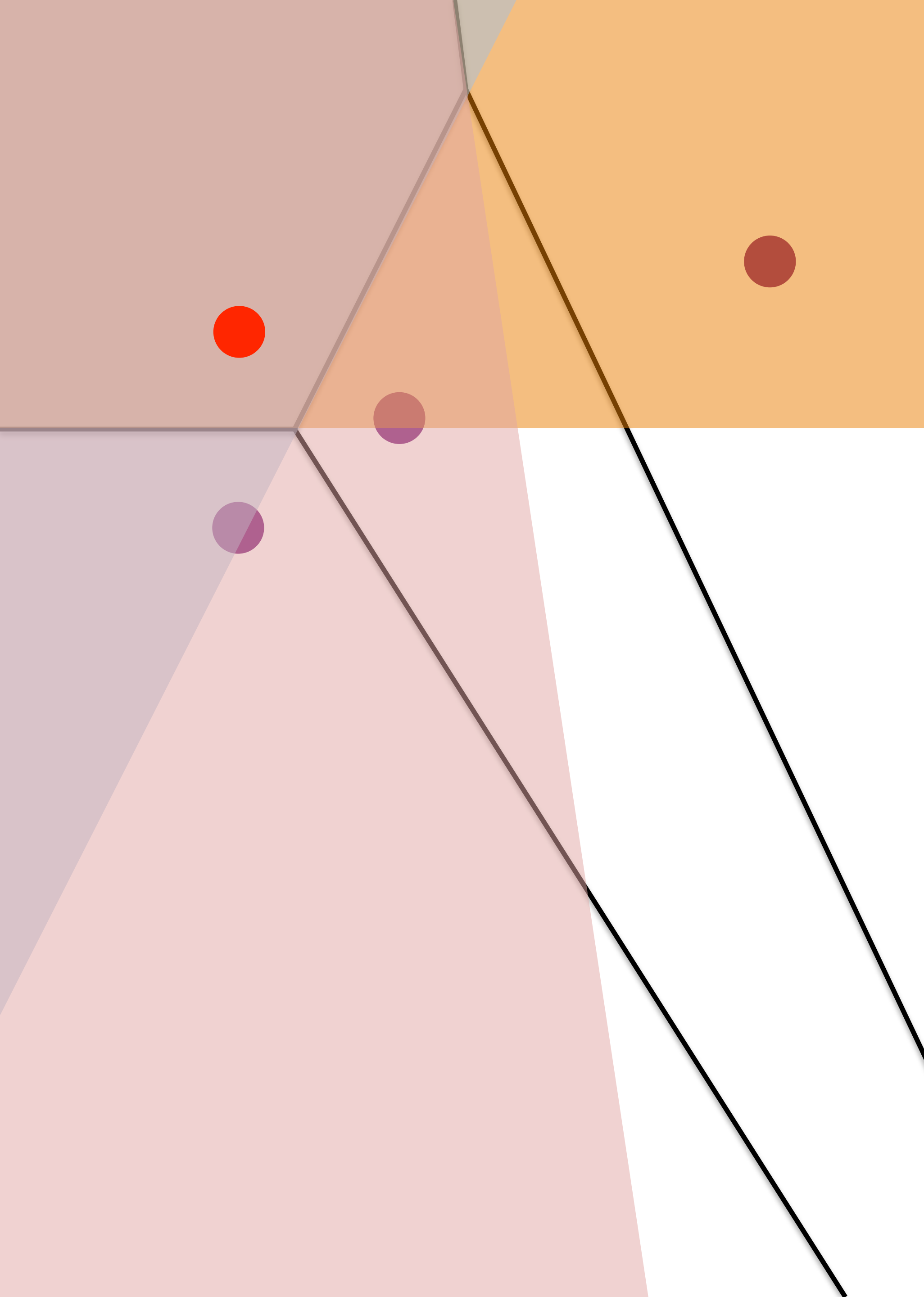

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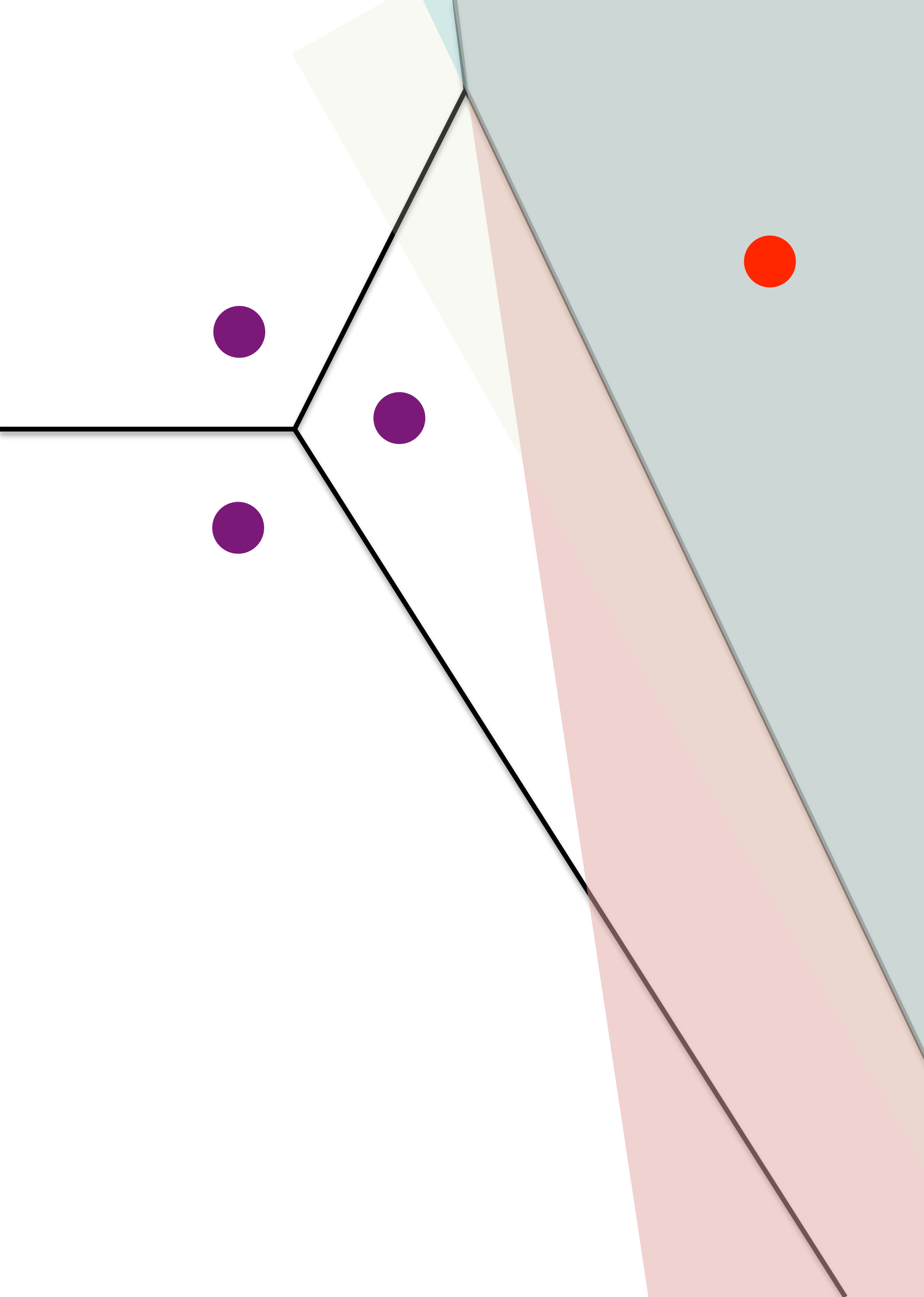

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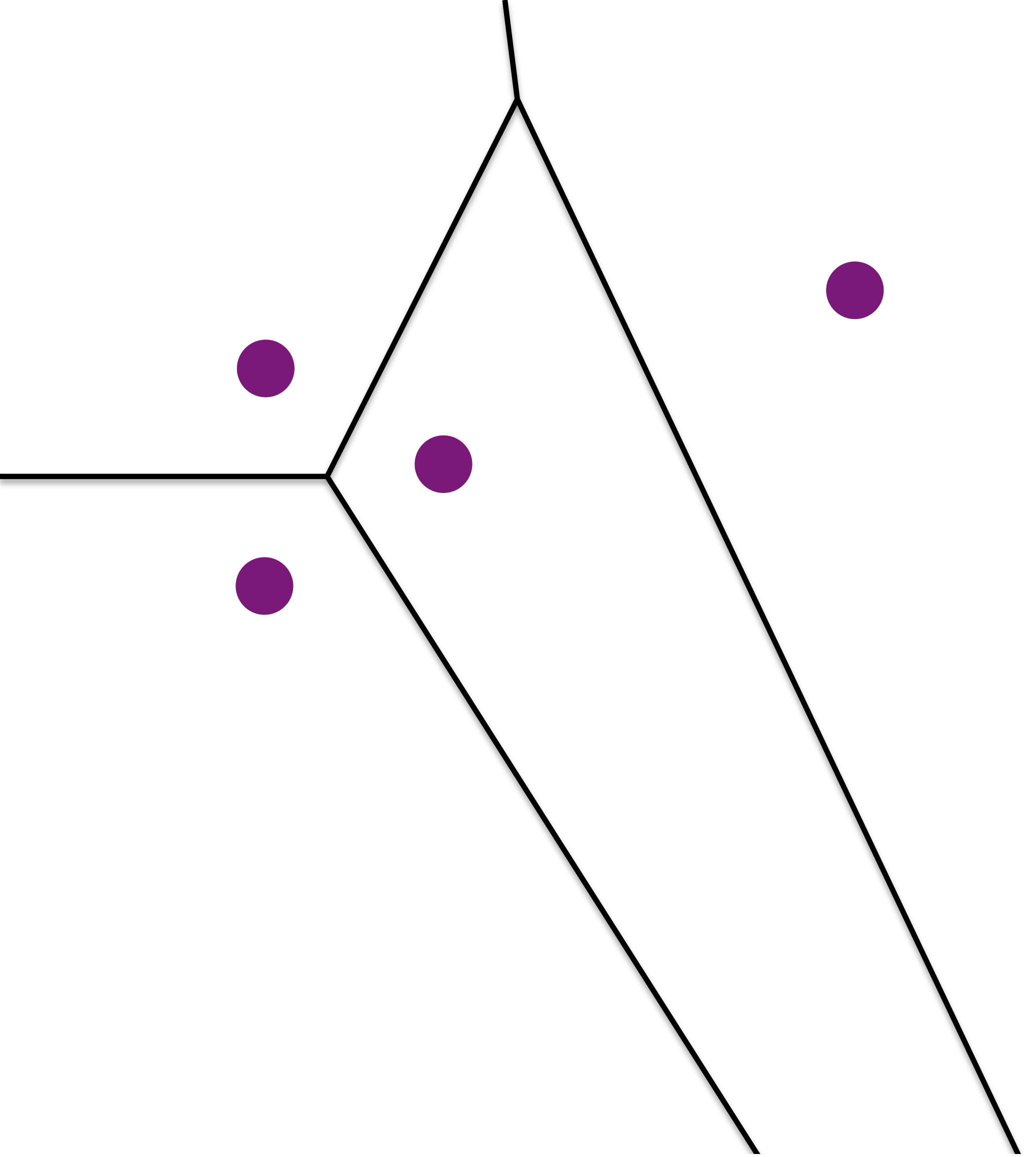

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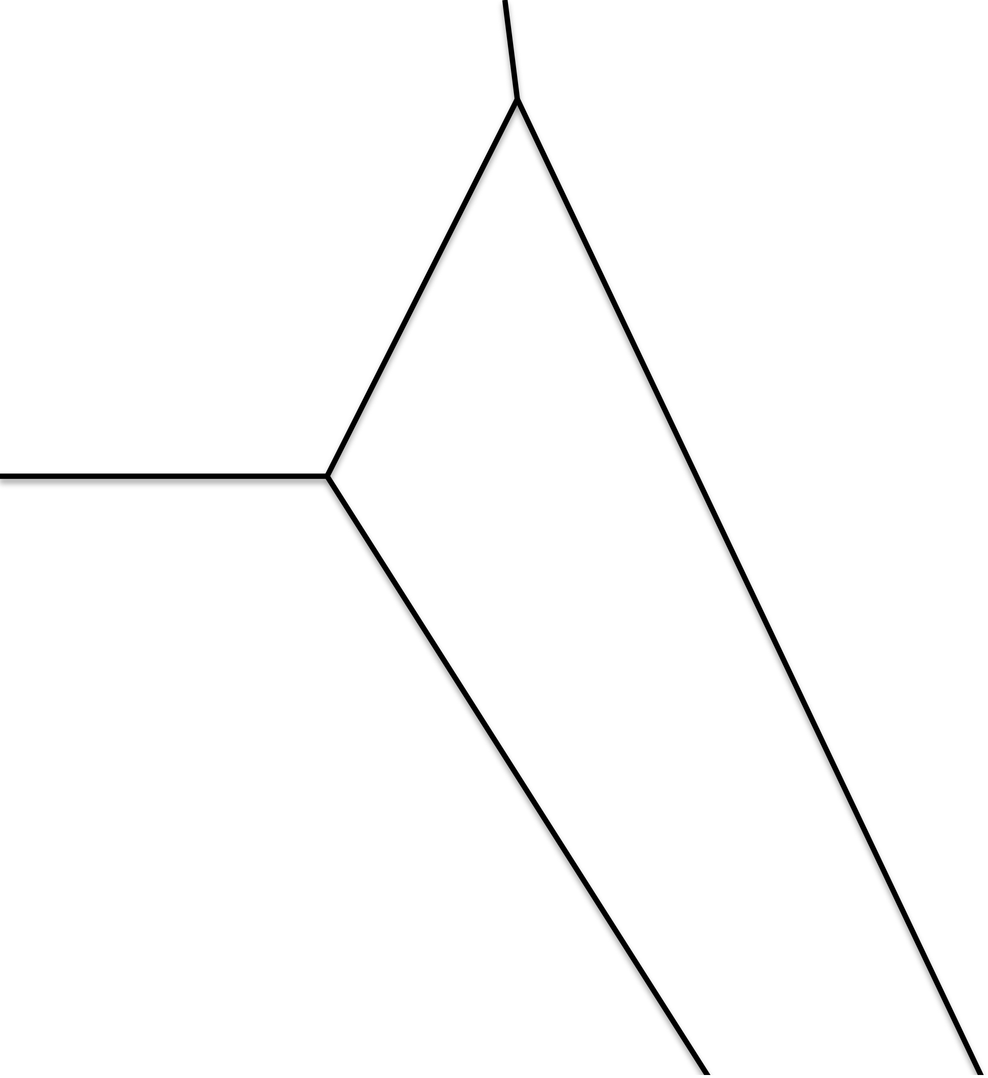

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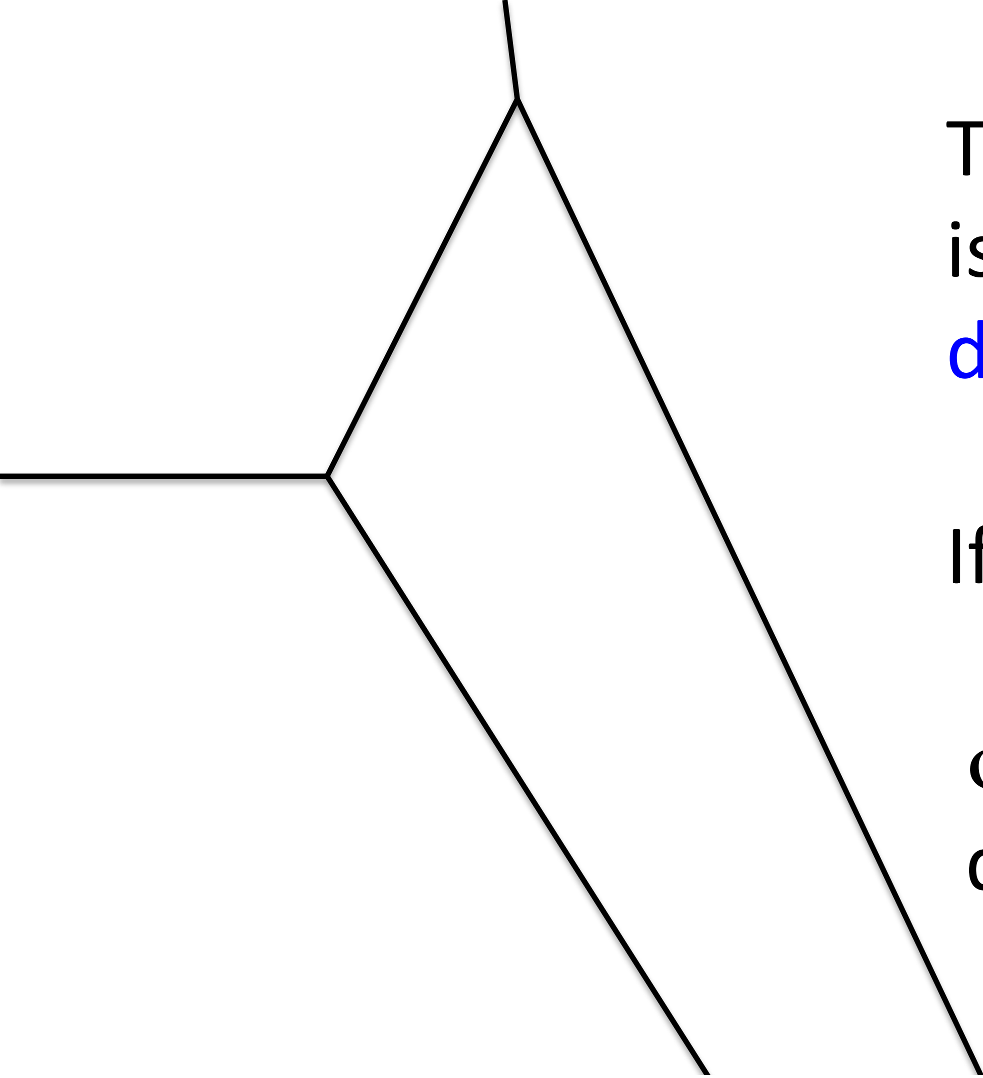

The *delaunay triangulation* is the dual to the voronoi diagram

If $\bigcap_{w \text{ in } \sigma} C_v \neq \emptyset$, then

σ is a simplex in the delaunay triangulation.

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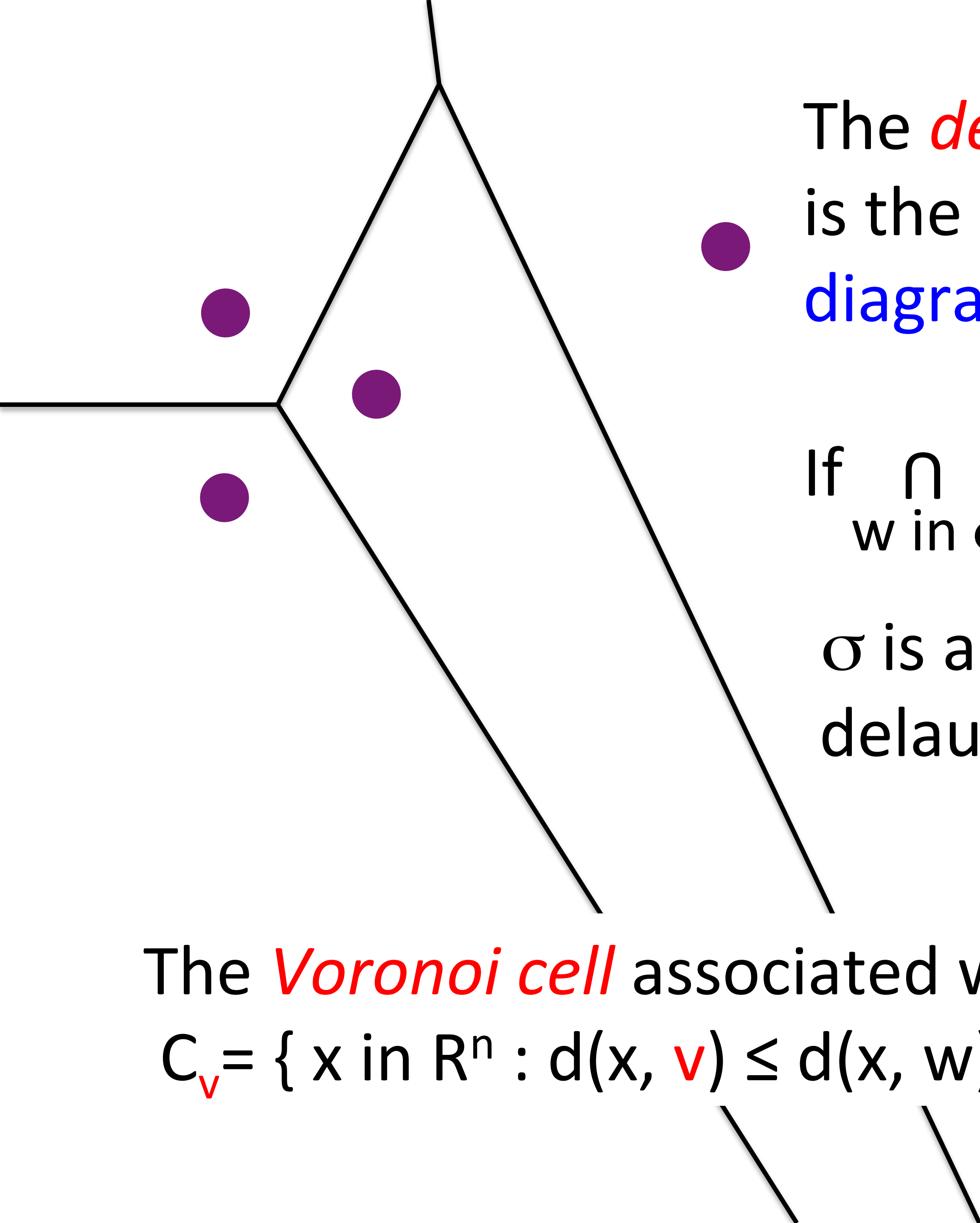

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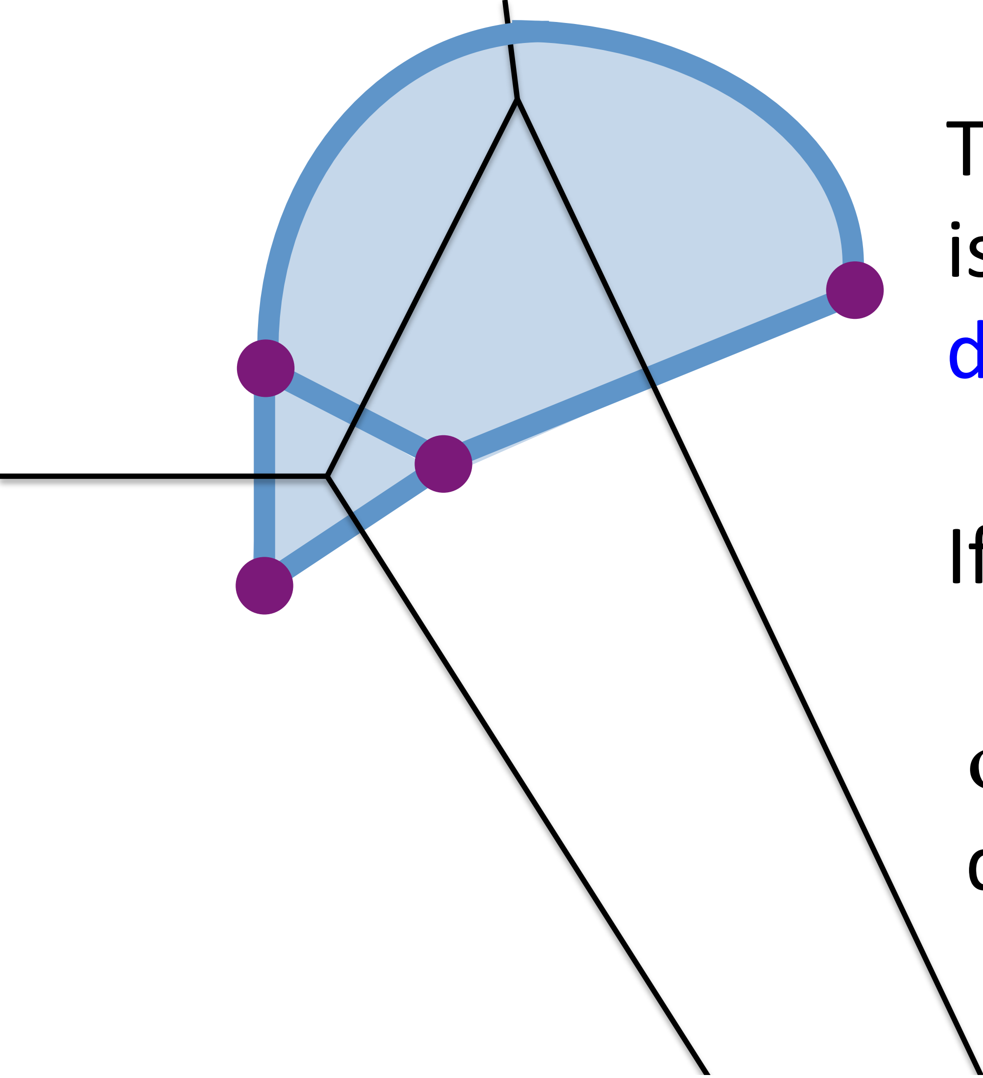

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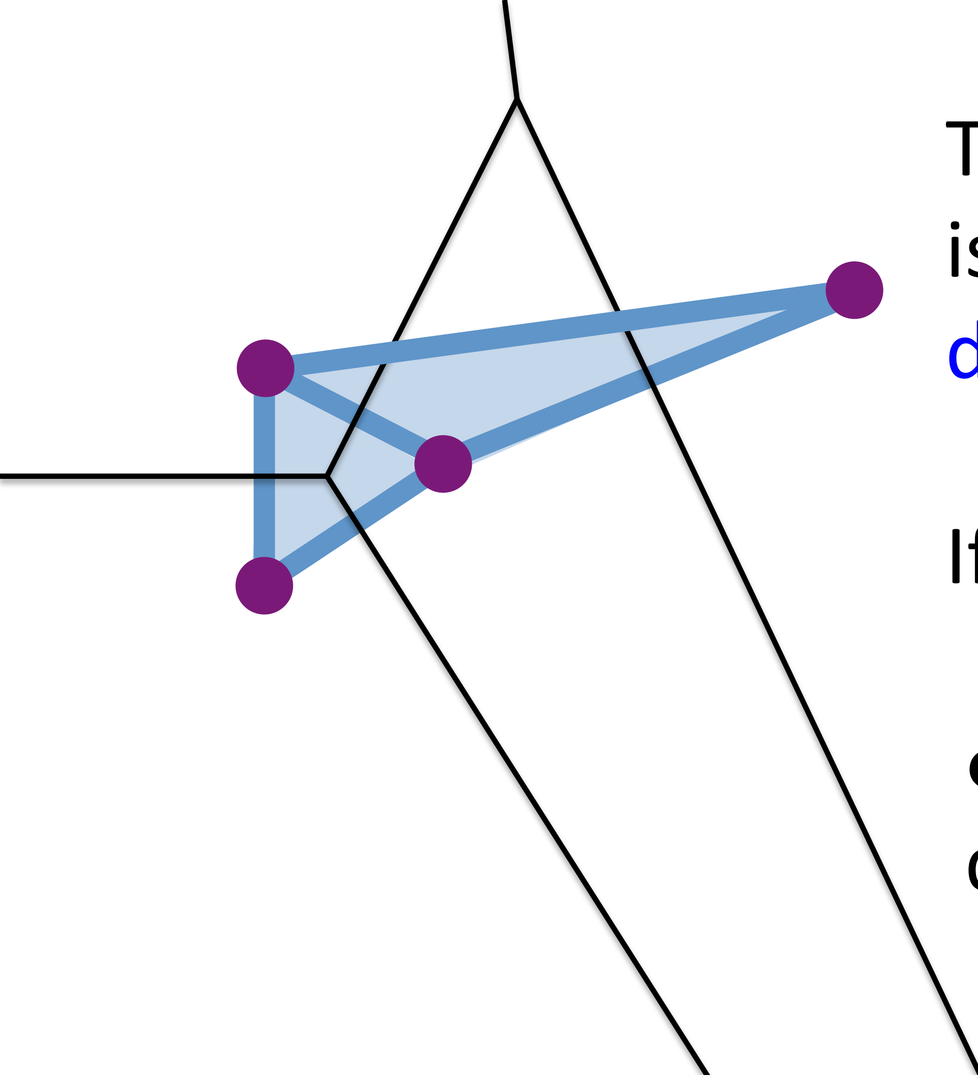

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