MATH:7450 (22M:305) Topics in Topology: Scientific and Engineering Applications of Algebraic Topology

Sept 20, 2013: Persistent homology.

Fall 2013 course offered through the
University of Iowa Division of Continuing Education

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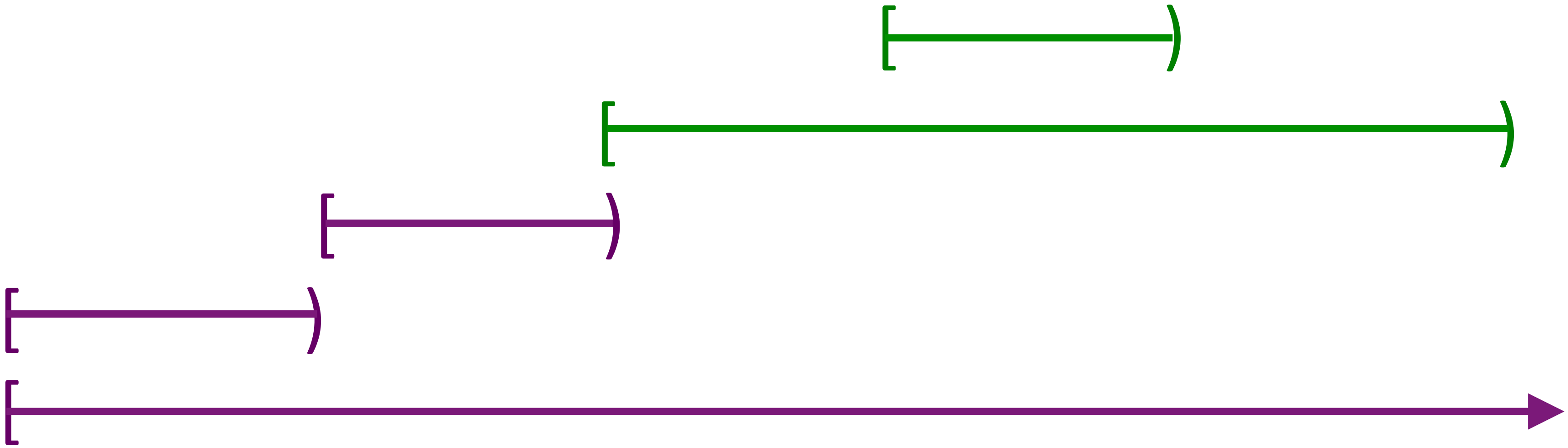
Applied Mathematical and Computational Sciences,
University of Iowa

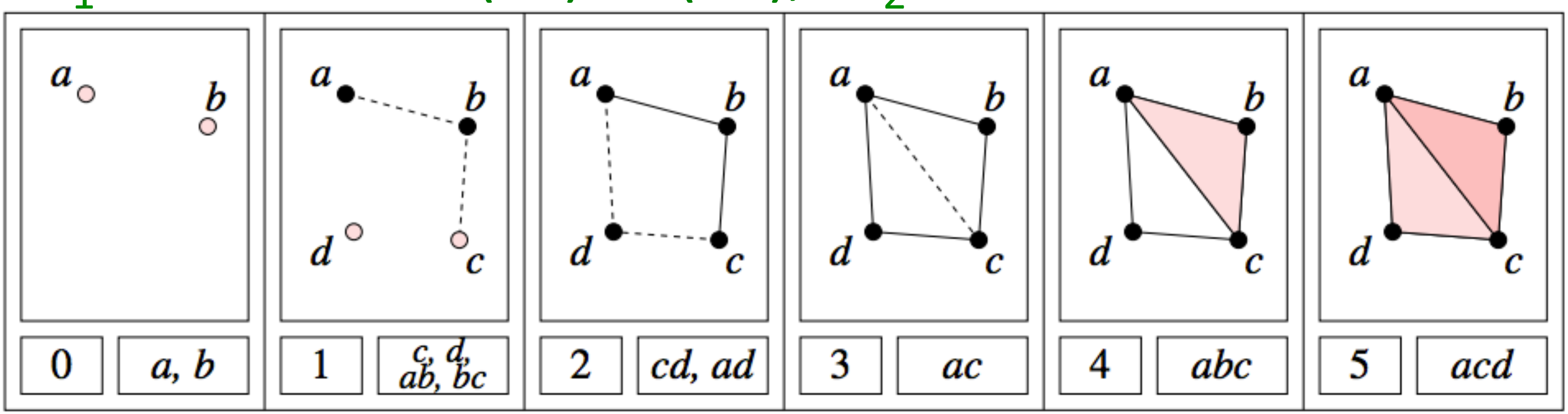
http://www.math.uiowa.edu/~idarcy/AppliedTopology.html

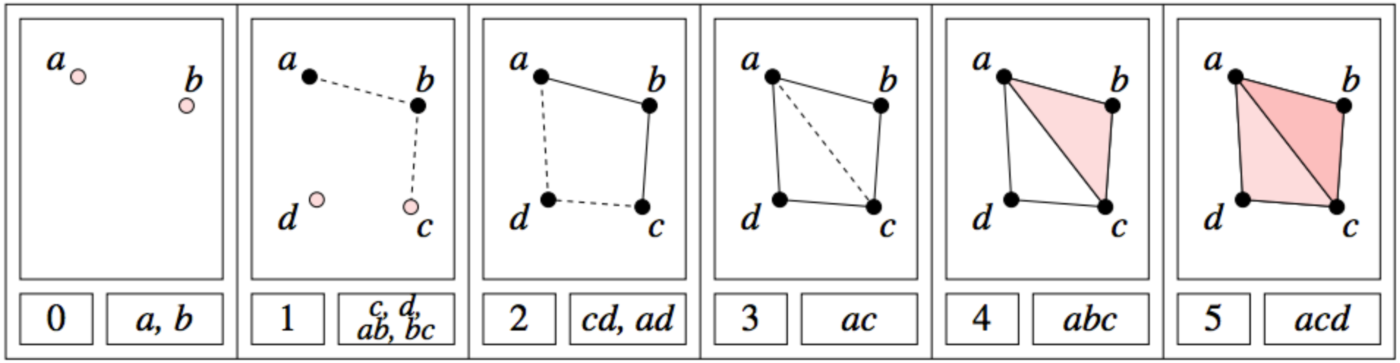
$$H_0 = \langle a, b, c, d \;:\; tc + td,\; tb + c,\; ta + tb \rangle$$

$$H_1 = \langle z_1, z_2 \;:\; t\,z_2,\; t^3 z_1 + t^2 z_2 \rangle$$



$$z_1 = ad + cd + t(bc) + t(ab), \quad z_2 = ac + t^2bc + t^2ab$$





$$
\begin{array}{ccccccc}
C_2^0 & \xrightarrow{f^0} & C_2^1 & \xrightarrow{f^1} & C_2^2 & \xrightarrow{f^2} & \cdots \\
\downarrow \partial_2 & & \downarrow \partial_2 & & \downarrow \partial_2 & & \\
C_1^0 & \xrightarrow{f^0} & C_1^1 & \xrightarrow{f^1} & C_1^2 & \xrightarrow{f^2} & \cdots \\
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\end{array}
$$

p-persistent k^{th} homology group:

$$H_k^{i,p} = Z_k^i / (B_k^{i+p} \cap Z_k^i)$$

http://link.springer.com/article/10.1007%2Fs00454-004-1146-y

$\partial_3 \downarrow$

$C_2 = \mathbf{Z_2}[abc, acd]$

$\partial_2 \downarrow$

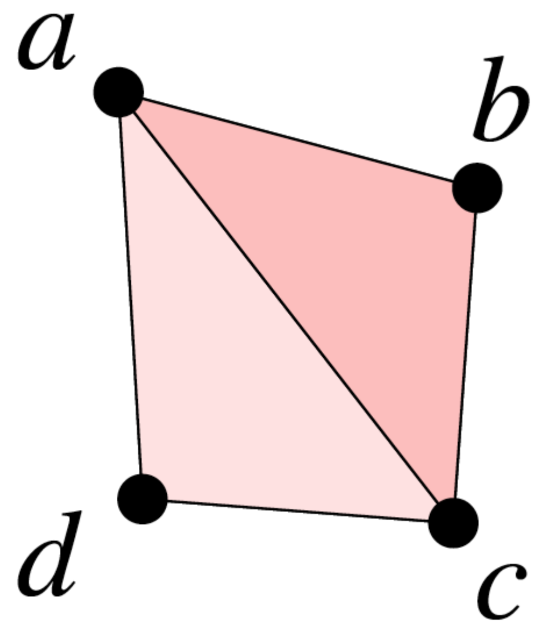
$C_1 = \mathbf{Z_2}[ab, bc, ac, ad, cd]$

$\partial_1 \downarrow$

$C_0 = \mathbf{Z_2}[a, b, c, d]$

$\partial_0 \downarrow$

0



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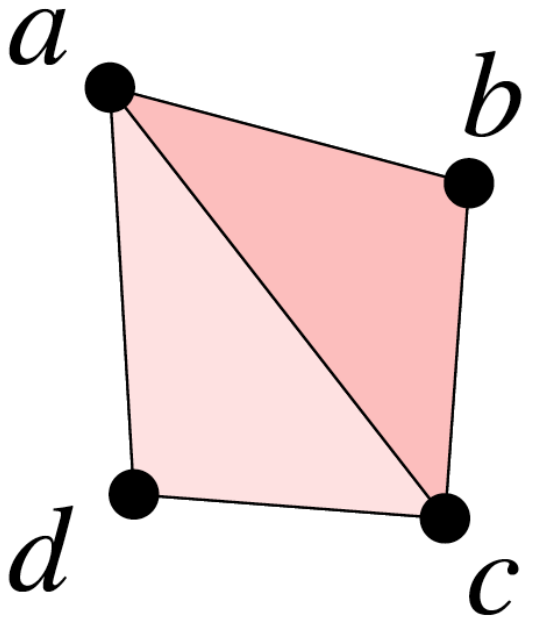

$\oplus C_k = C_0 \oplus C_1 \oplus C_2 \oplus C_3 \oplus \ldots$

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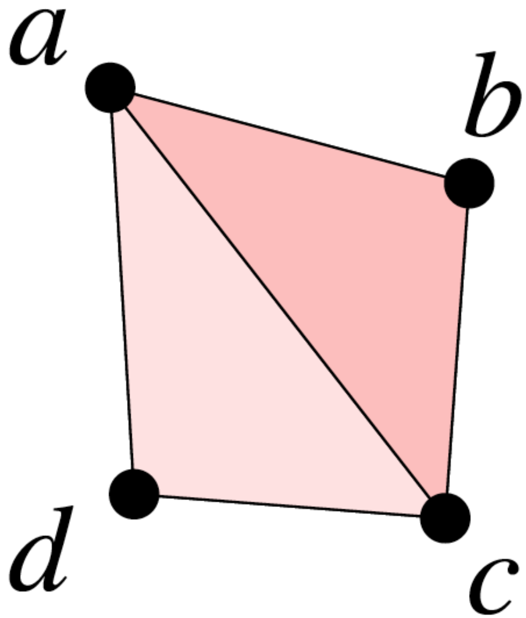
0



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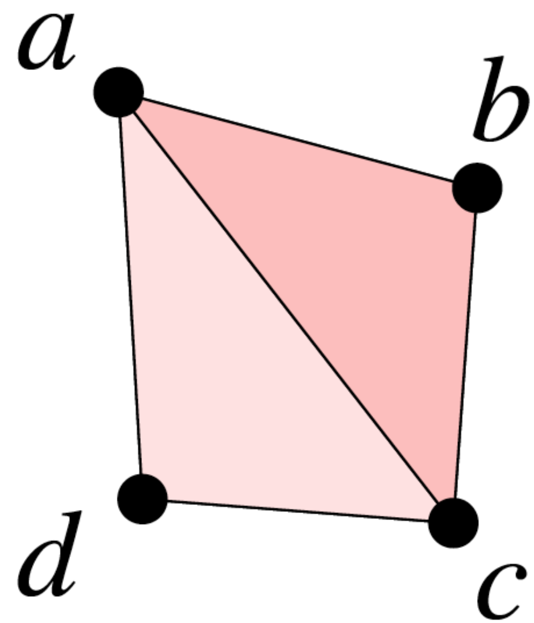

$\oplus C_k = C_0 \oplus C_1 \oplus C_2$

$a + b + bc + abc + acd$

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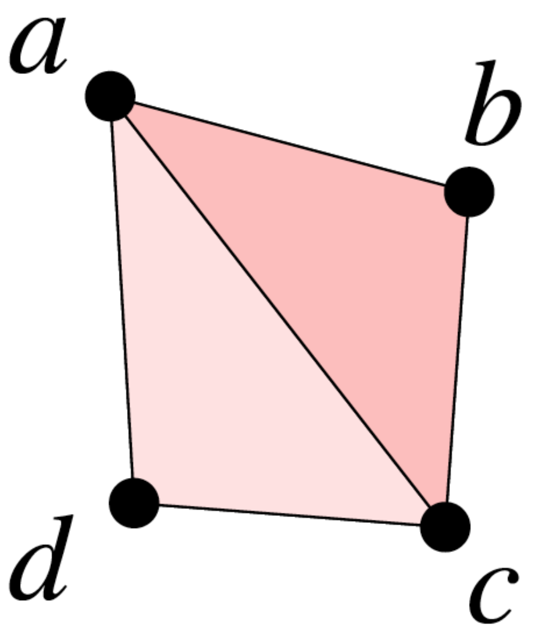
$\partial_0 \downarrow$

0



$\oplus C_k = C_0 \oplus C_1 \oplus C_2$

$a + b \oplus bc \oplus abc + acd$

$(a + b, \; bc, \; abc + acd)$

$\partial_3 \downarrow$

$C_2 = \mathbf{Z_2}[abc, acd]$

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$C_1 = \mathbf{Z_2}[ab, bc, ac, ad, cd]$

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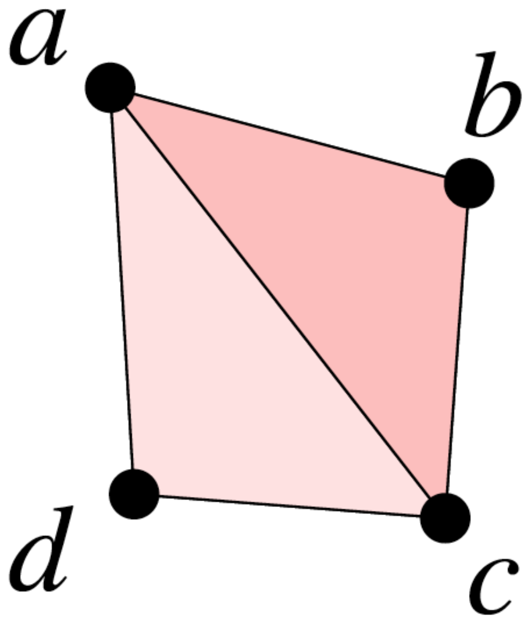
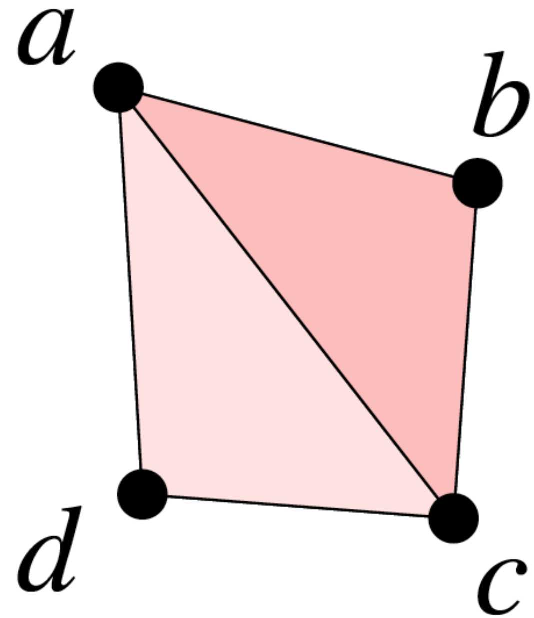


$\oplus C_k = C_0 \oplus C_1 \oplus C_2 \oplus C_3 \oplus \ldots$

$\partial : \oplus C_k \rightarrow \oplus C_k$

$\partial(0, ab, 0) = (a + b, 0, 0)$

$\partial^2 = 0$



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$\partial_0 \downarrow$

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$$\partial : \oplus C_k \rightarrow \oplus C_k$$

$$\partial(0, ab, 0) = (a + b, 0, 0)$$

$$\partial(0 \oplus ab \oplus 0) = a{+}b \oplus 0 \oplus 0$$

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$C_0 = \mathbf{Z_2}[a, b, c, d]$

$\partial_0 \downarrow$

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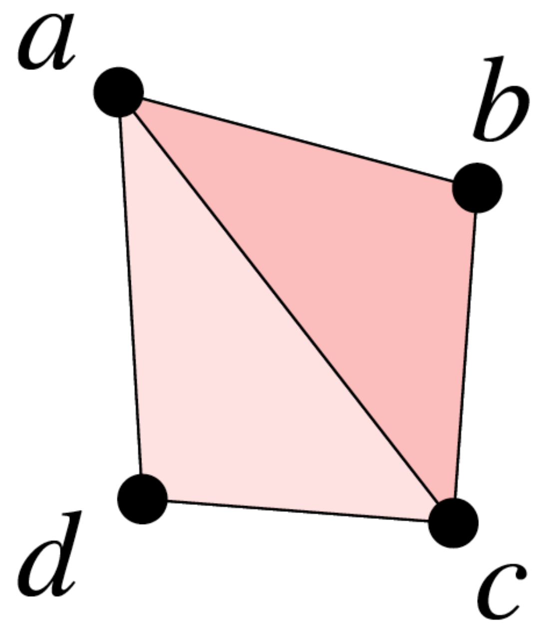

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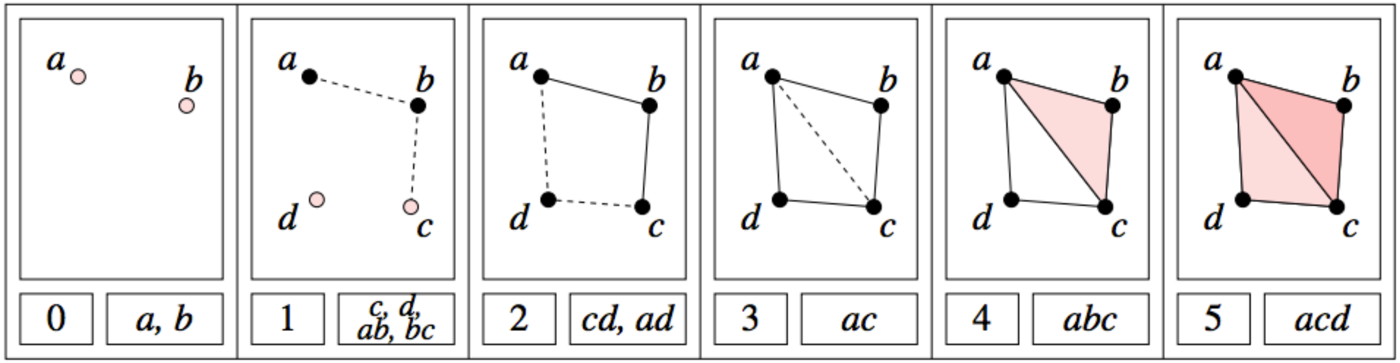

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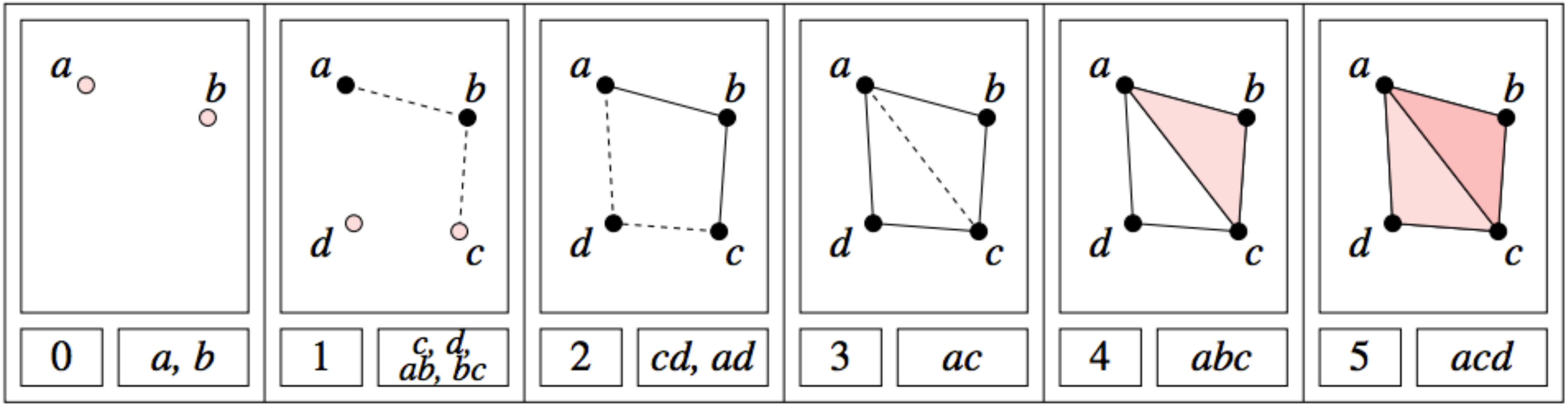
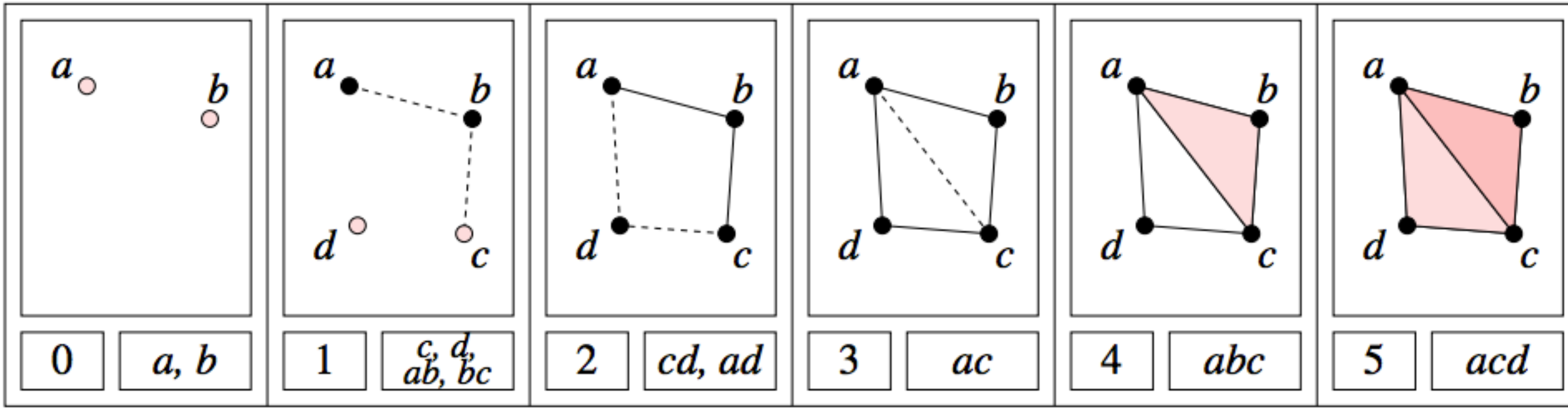


$$C_j^0 \rightarrow C_j^1 \rightarrow C_j^2 \rightarrow C_j^3 \rightarrow C_j^4 \rightarrow C_j^5$$



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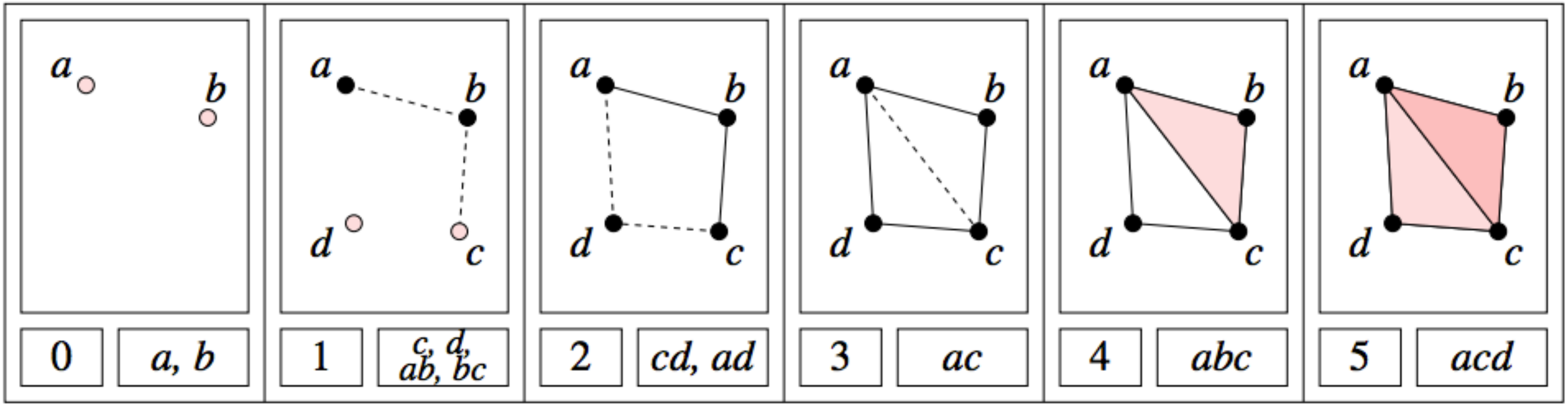
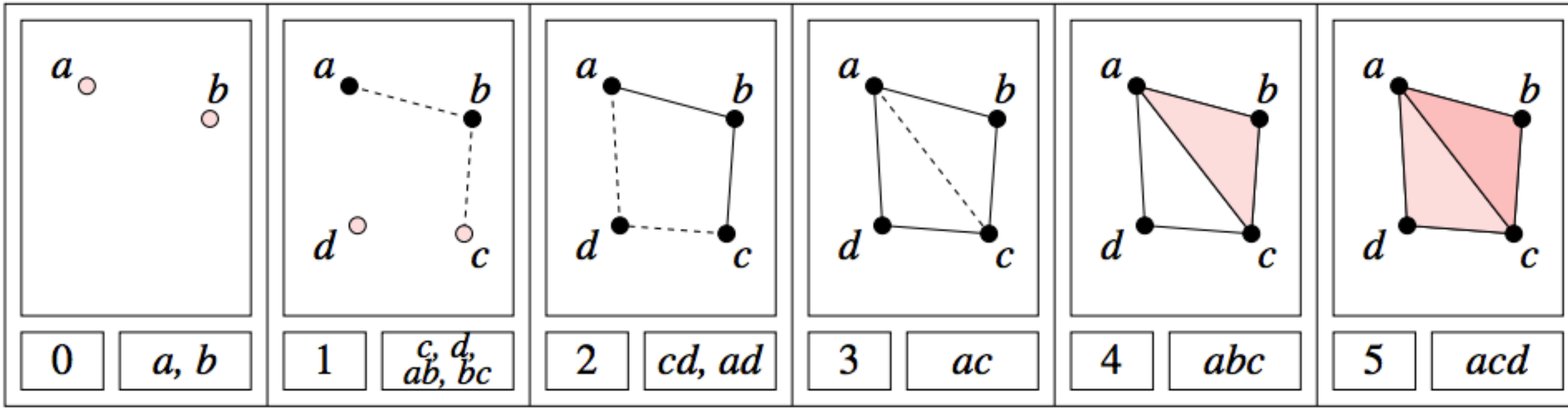


$$C_0^0 \rightarrow C_0^1 \rightarrow C_0^2 \rightarrow C_0^3 \rightarrow C_0^4 \rightarrow C_0^5 \rightarrow \ldots$$

$$\oplus C_0^i = \langle a, b\rangle \oplus \langle c, d, ta, tb\rangle \oplus \langle tc, td, t^2a, t^2b\rangle \oplus \ldots$$

$$(a, c + ta, tc + t^2b, t^2c, t^4b, t^5b, t^6b, \ldots)$$



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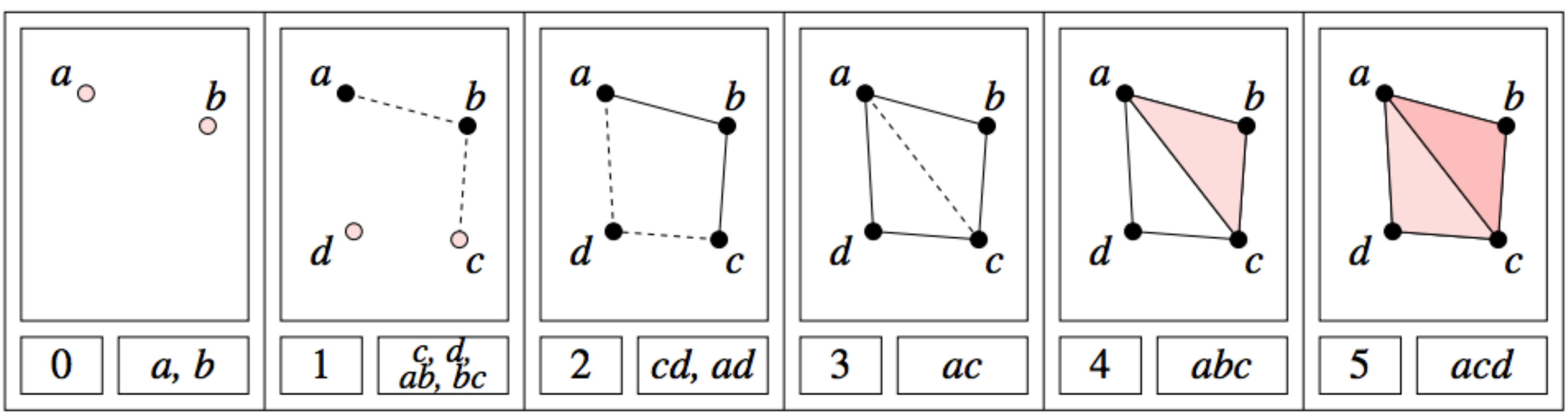
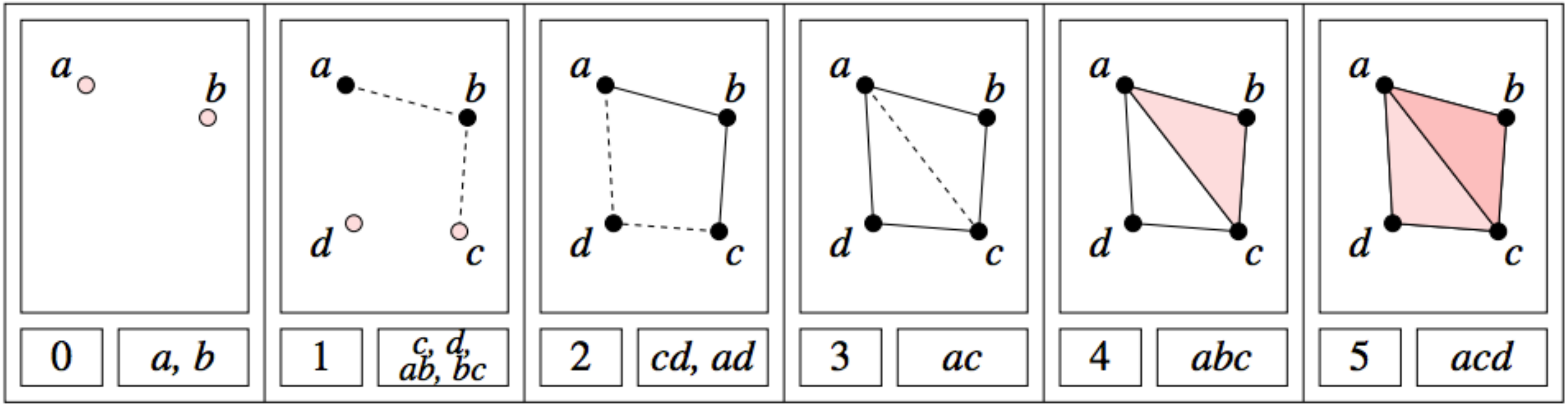
$$(a, c + ta, tc + t^2b, t^2c, t^4b, t^5b, t^6b, \ldots)$$

$$t \bullet (a, c + ta, tc + t^2b, t^2c, t^4b, t^5b, t^6b, \ldots)$$

$$= (0, ta, tc + t^2a, t^2c + t^3b, t^3c, t^5b, t^6b, t^7b, \ldots)$$

$$H_0 = \langle a, b, c, d : tc + td,\ tb + c,\ ta + tb \rangle$$





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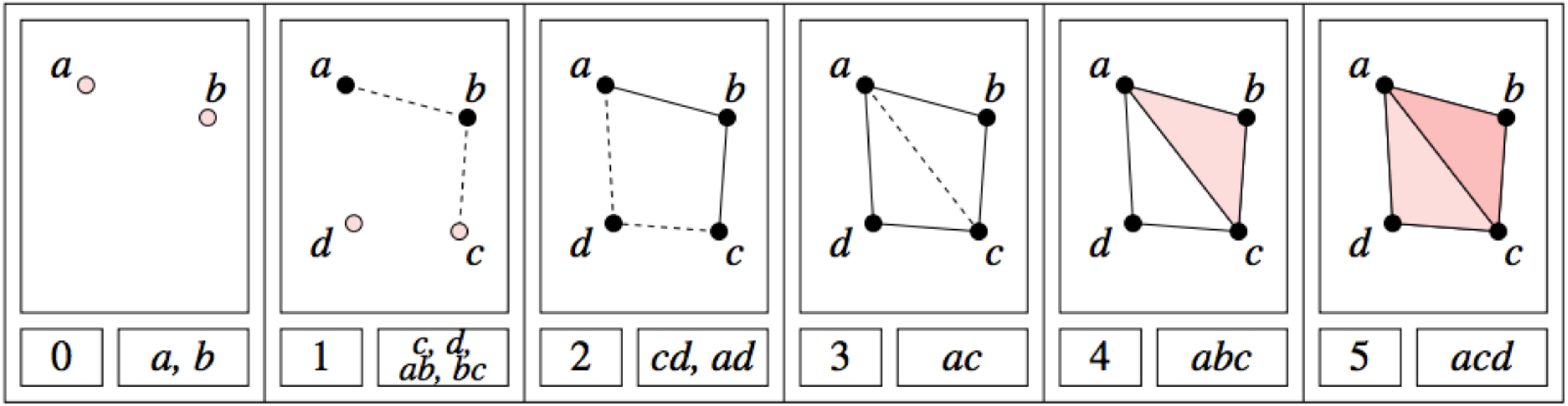
a: $\mathbf{Z_2}[t]$ b: $\mathbf{Z_2}[t]$ c: $\Sigma^1\, \mathbf{Z_2}[t]$ d: $\Sigma^1\, \mathbf{Z_2}[t]$



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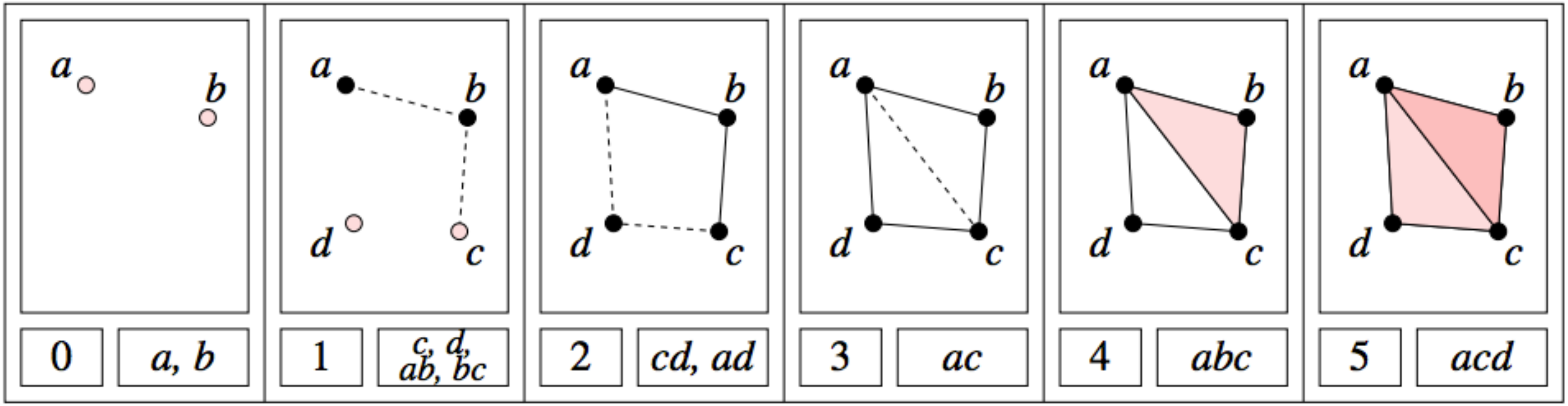
a: $\mathbf{Z_2}[t]$ b: $\mathbf{Z_2}[t]$ c: $\Sigma^1 \mathbf{Z_2}[t]$ d: $\Sigma^1 \mathbf{Z_2}[t]$

acd: $\Sigma^5 \mathbf{Z_2}[t]$



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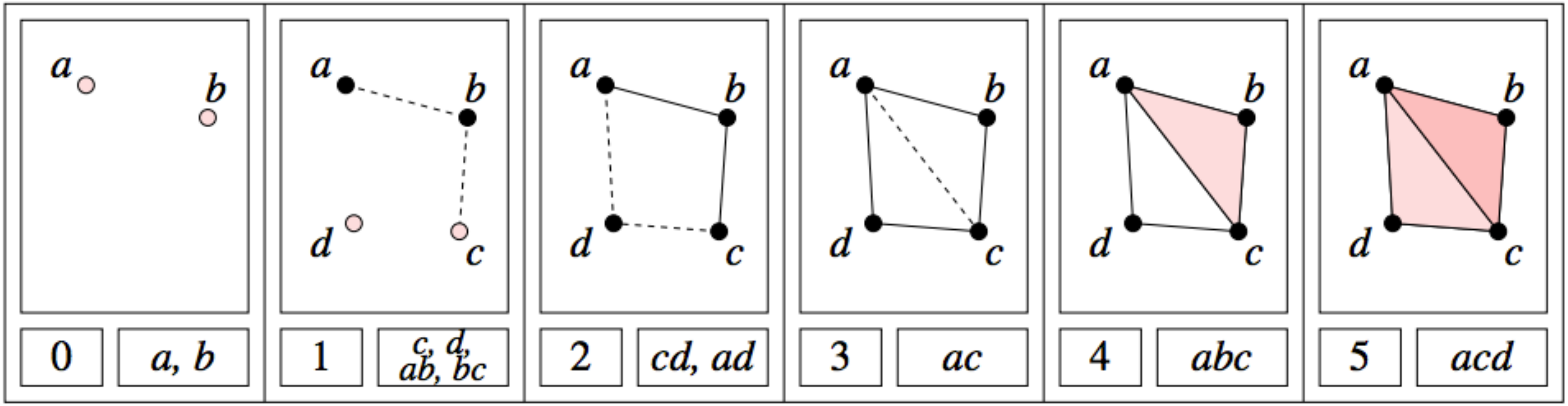
$$\text{a:} \quad \mathbf{Z_2}[t] = \{n_0 + n_1 t + n_2 t^2 + \ldots + n_k t^k : n_i \text{ in } \mathbf{Z_2}, k \text{ in } \mathbf{Z_+}\}$$

$$= \{ (n_0, n_1, n_2, \ldots, n_k, 0, 0, \ldots) : n_i \text{ in } \mathbf{Z_2}, k \text{ in } \mathbf{Z_+}\}$$



$$C_0^0 \rightarrow C_0^1 \rightarrow C_0^2 \rightarrow C_0^3 \rightarrow C_0^4 \rightarrow C_0^5 \rightarrow \dots$$

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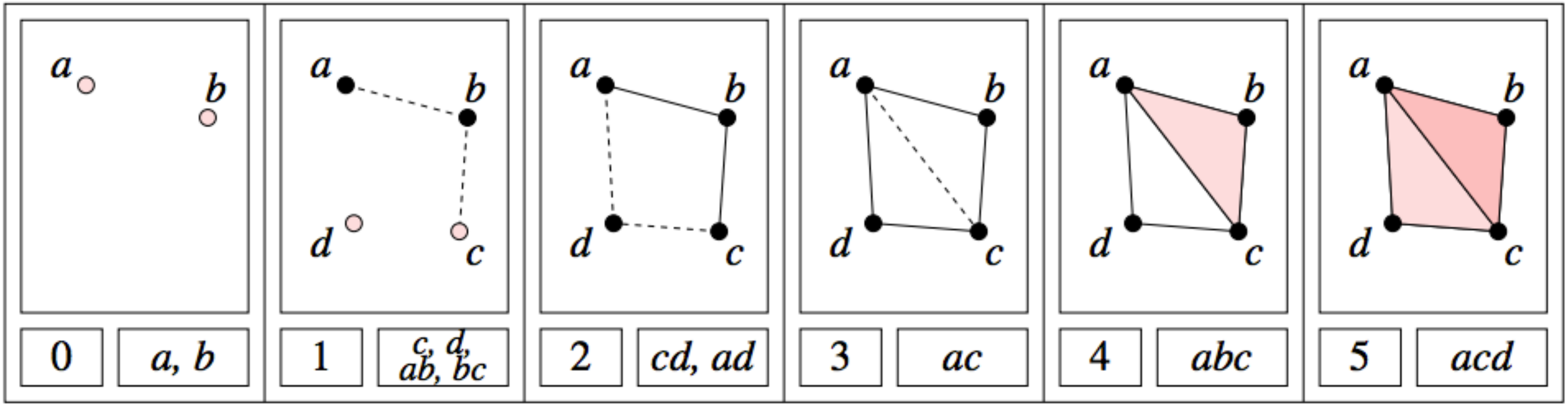
b: $\mathbf{Z_2}[t] = \{n_0 + n_1 t + n_2 t^2 + \dots + n_k t^k : n_i \text{ in } \mathbf{Z_2}, k \text{ in } \mathbf{Z_+}\}$

$= \{ (n_0, n_1, n_2, \dots, n_k, 0, 0, \dots) : n_i \text{ in } \mathbf{Z_2}, k \text{ in } \mathbf{Z_+}\}$



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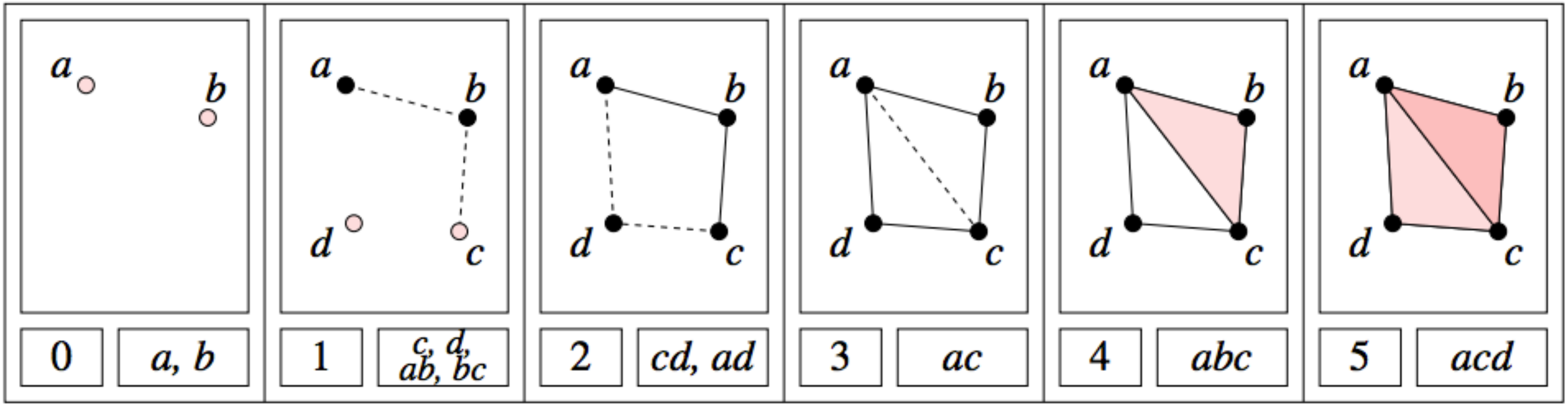
$$c: \quad \Sigma^1 \mathbf{Z_2}[t] \quad = \{n_0 t + n_1 t^2 + \ldots \quad : \ n_i \ \text{in} \ \mathbf{Z_2}\}$$

$$= \{ (0, n_0, n_1, \ldots) \quad : \ n_i \ \text{in} \ \mathbf{Z_2}\}$$



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$$d: \quad \Sigma^1 \mathbf{Z_2}[t] = \{n_0 t + n_1 t^2 + \ldots \; : \; n_i \text{ in } \mathbf{Z_2}\}$$

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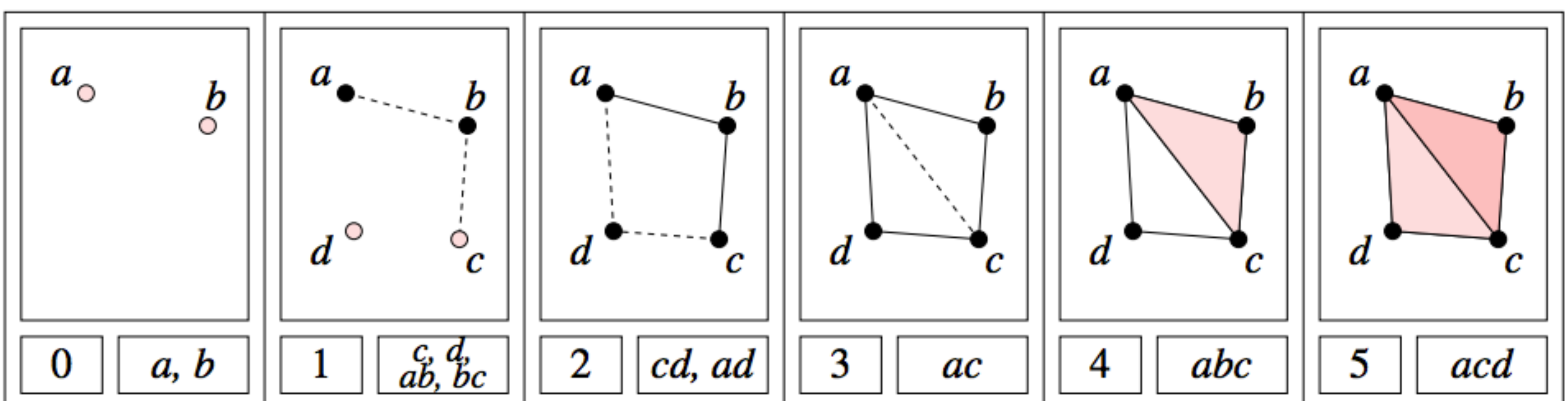
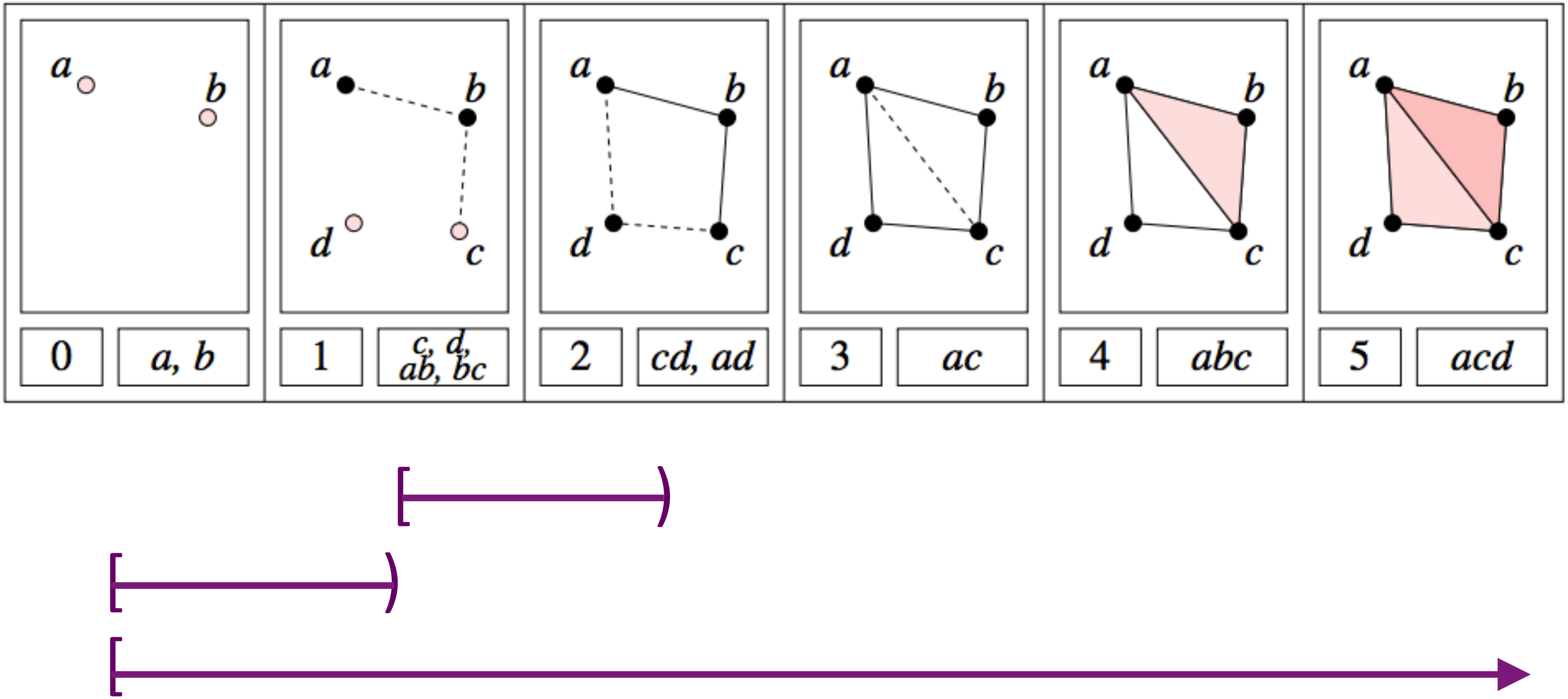
$$H_0 = < a, b, c, d \; : \; tc + td, \; tb + c, \; ta + tb >$$





$$H_0 = < a, b, c, d : tc + td, \; tb + c, \; ta + tb >$$

$$a: \quad \mathbf{Z_2}[t] = \{n_0 + n_1 t + n_2 t^2 + \ldots + n_k t^k : n_i \text{ in } \mathbf{Z_2}, k \text{ in } \mathbf{Z_+}\}$$

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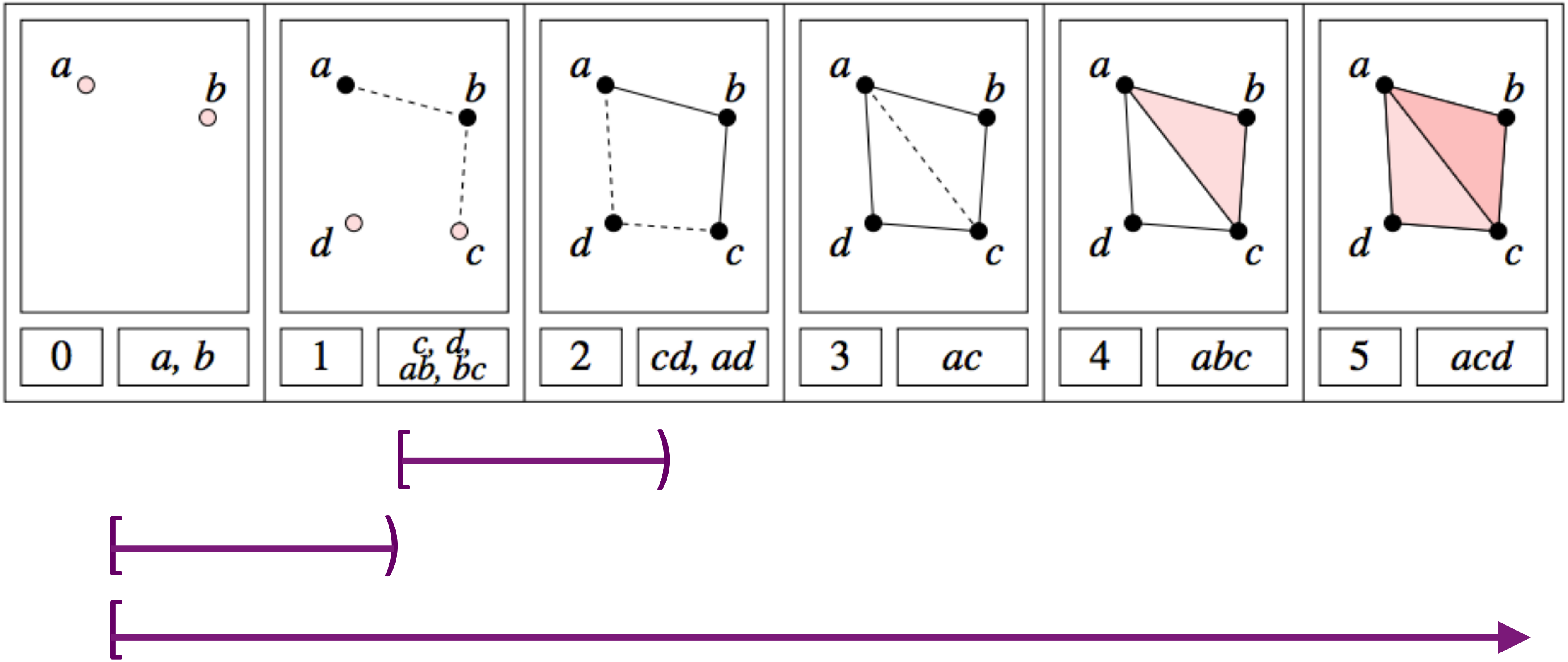
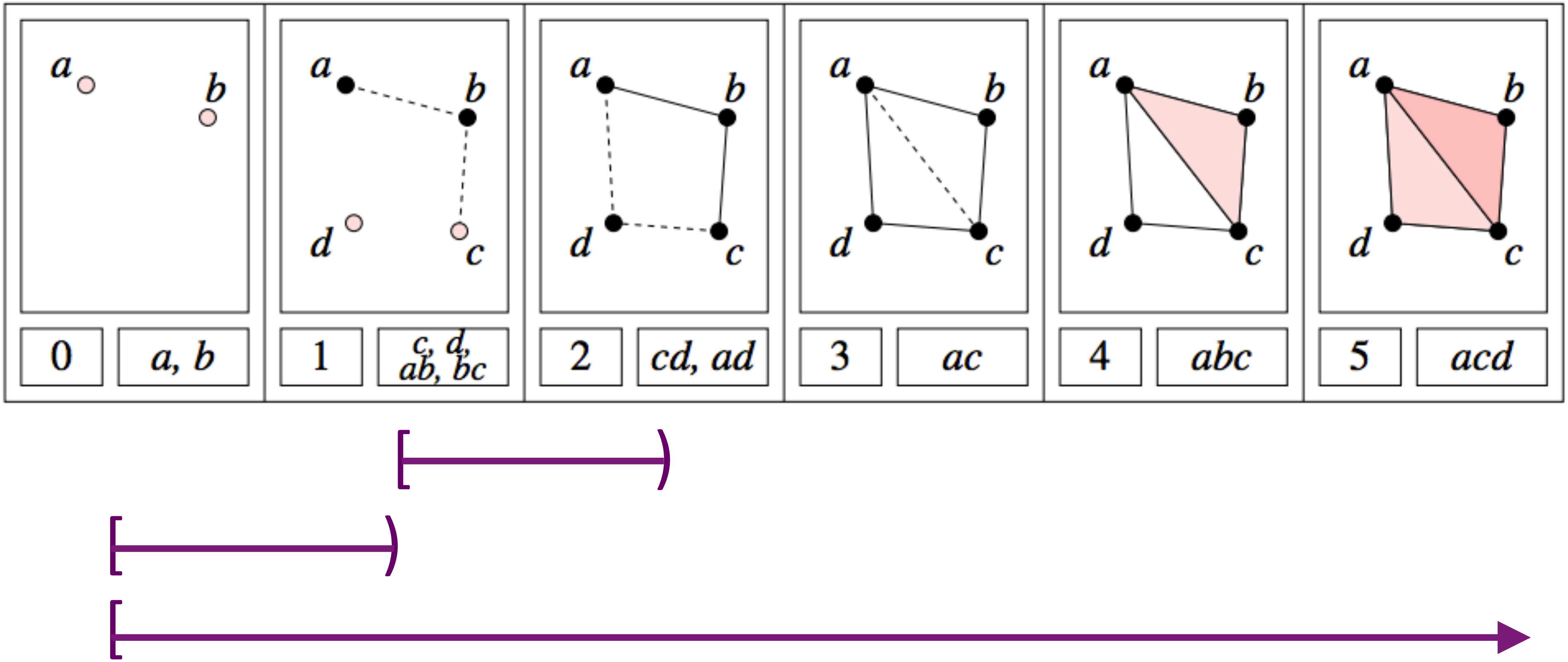


$$H_0 = \langle a, b, c, d : tc + td, \; tb + c, \; ta + tb \rangle$$

$$\mathbf{Z_2}[t] = \{n_0 + n_1 t + n_2 t^2 + \dots : n_i \text{ in } \mathbf{Z_2}\}$$

b: $\mathbf{Z_2}[t]/t = \{n_0 : n_0 \text{ in } \mathbf{Z_2}\} = \mathbf{Z_2}$

$$= \{ (n_0, 0, 0, 0, \dots) : n_i \text{ in } \mathbf{Z_2}\}$$

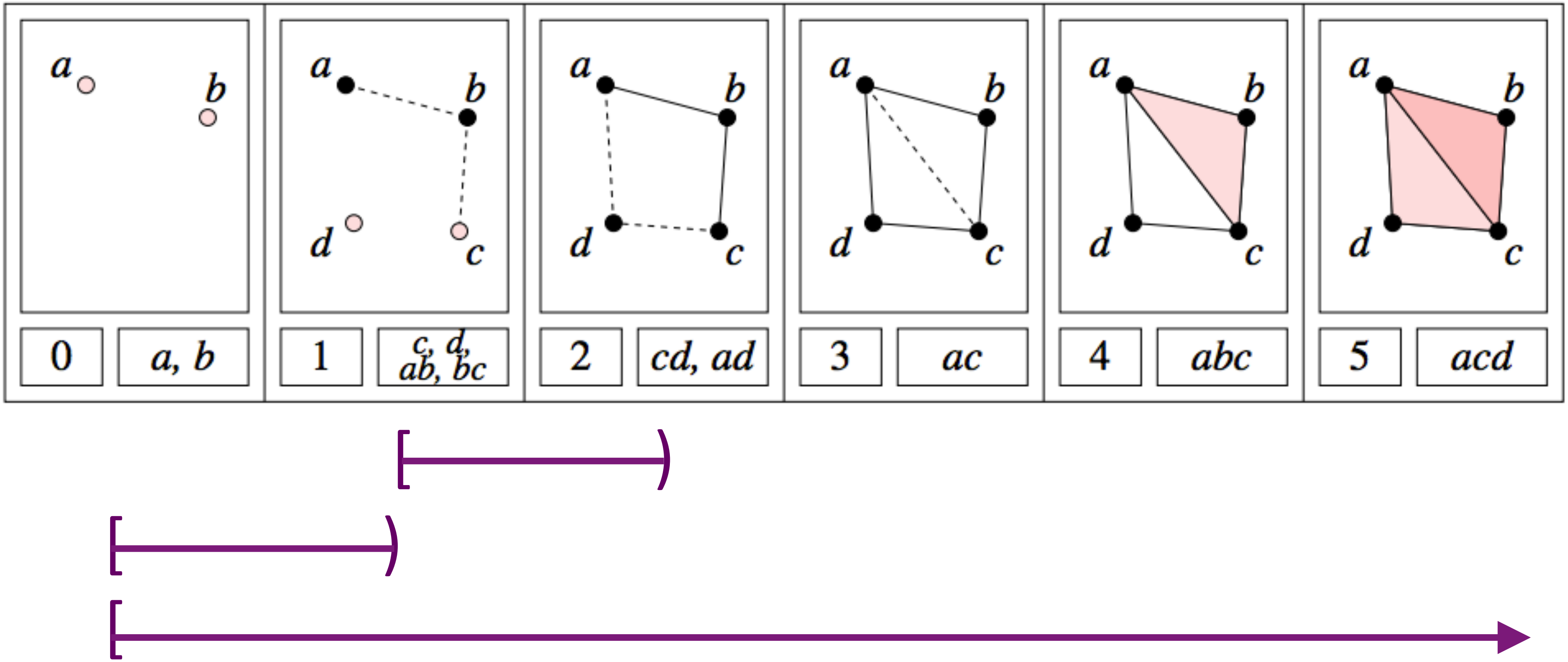


$$H_0 = \langle a, b, c, d \; : \; tc + td, \; tb + c, \; ta + tb \rangle$$

$$\mathbf{Z_2}[t] = \{n_0 + n_1 t + n_2 t^2 + \ldots \; : \; n_i \text{ in } \mathbf{Z_2}\}$$

$$d: \; \Sigma^1 \mathbf{Z_2}[t]/t = \Sigma^1 \{n_0 \; : \; n_0 \text{ in } \mathbf{Z_2}\}$$

$$= \{ (0, n_0, 0, 0, 0, \ldots) : \; n_0 \text{ in } \mathbf{Z_2}\}$$



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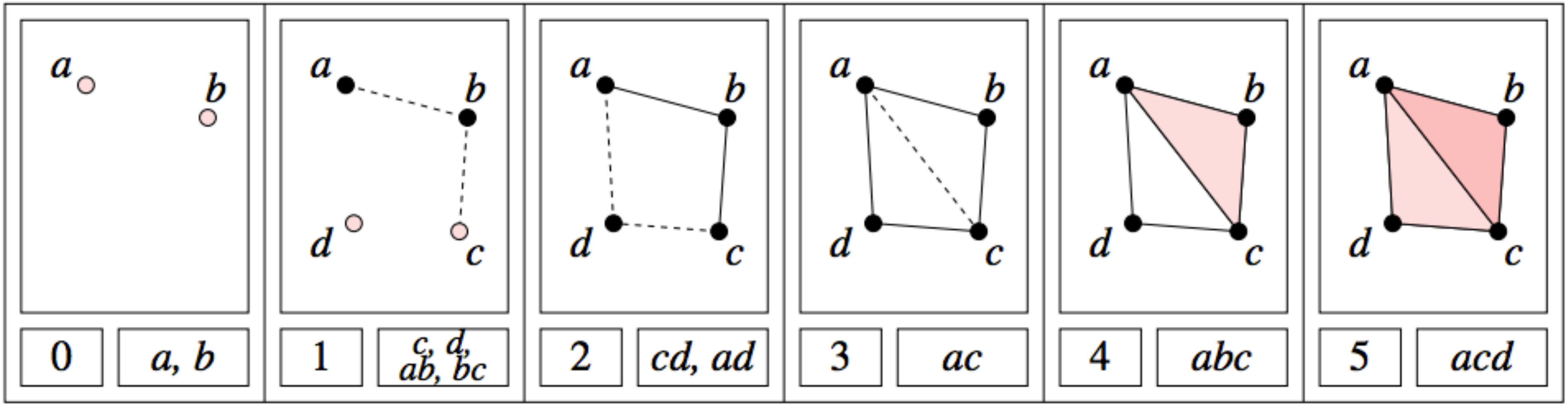
$$c: \;\; \Sigma^1 \mathbf{Z_2}[t]/1 = \{ (0, 0, 0, \ldots) \}$$



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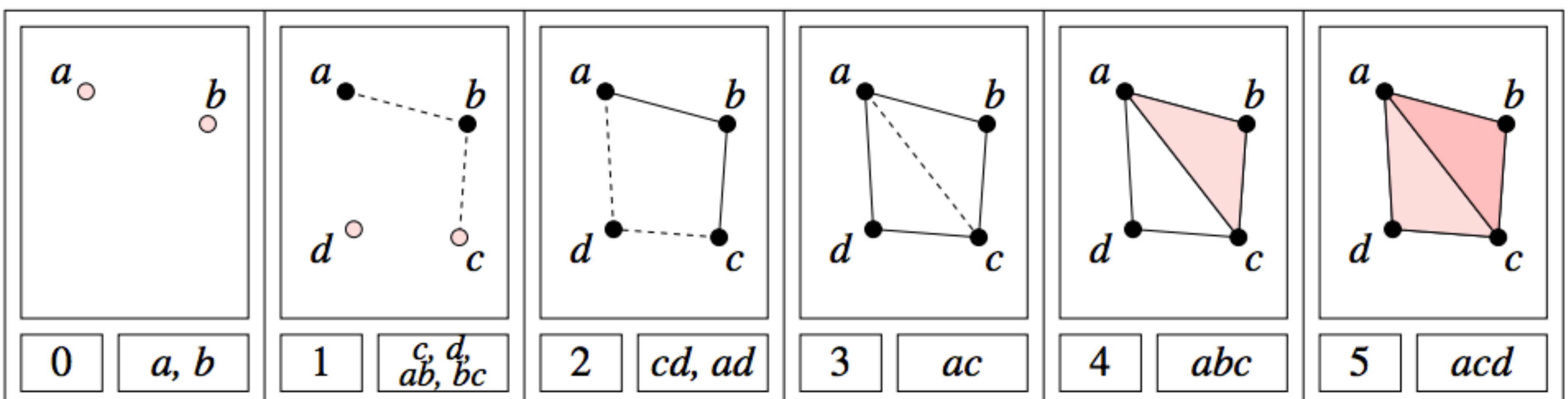
b: $\mathbf{Z_2}[t]/t = \{n_0 : n_0 \text{ in } \mathbf{Z_2}\} = \mathbf{Z_2}$

c: $\Sigma^1 \mathbf{Z_2}[t]/1 = \text{empty set}$

d: $\Sigma^1 \mathbf{Z_2}[t]/t = \{n_0 : n_0 \text{ in } \mathbf{Z_2}\} = \{(0, n_0, 0, 0, 0, \ldots)\}$

$$H_0 = < a, b, c, d \; : \; tc + td, \; tb + c, \; ta + tb >$$

$$= \mathbf{Z_2}[t] \oplus \mathbf{Z_2}[t]/t \oplus (\Sigma^1 \mathbf{Z_2}[t])/t$$



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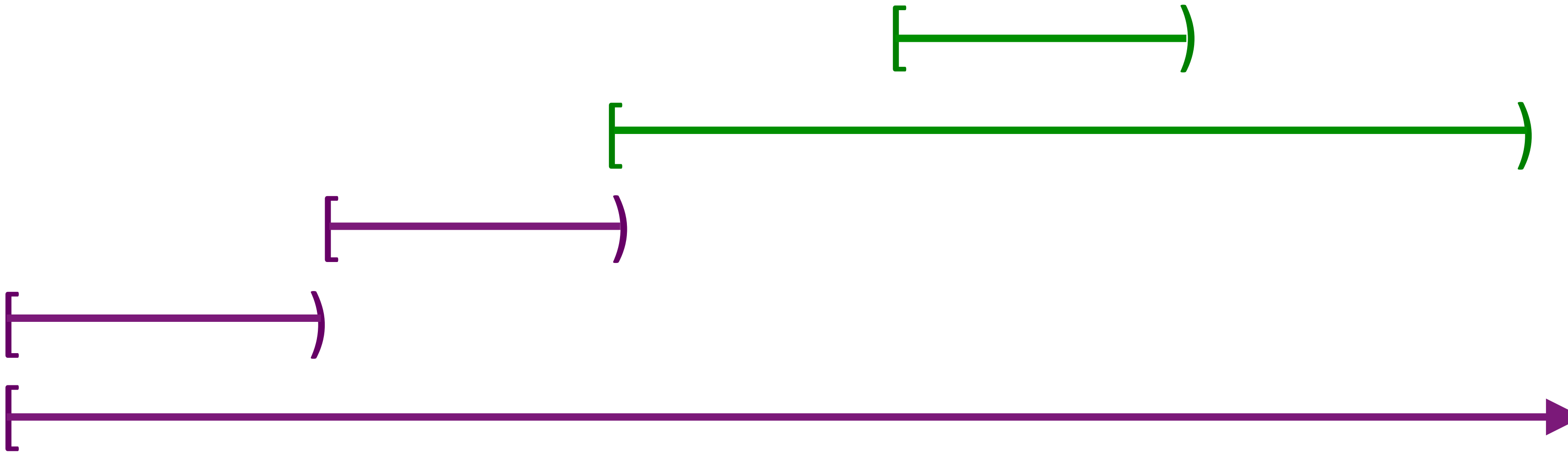
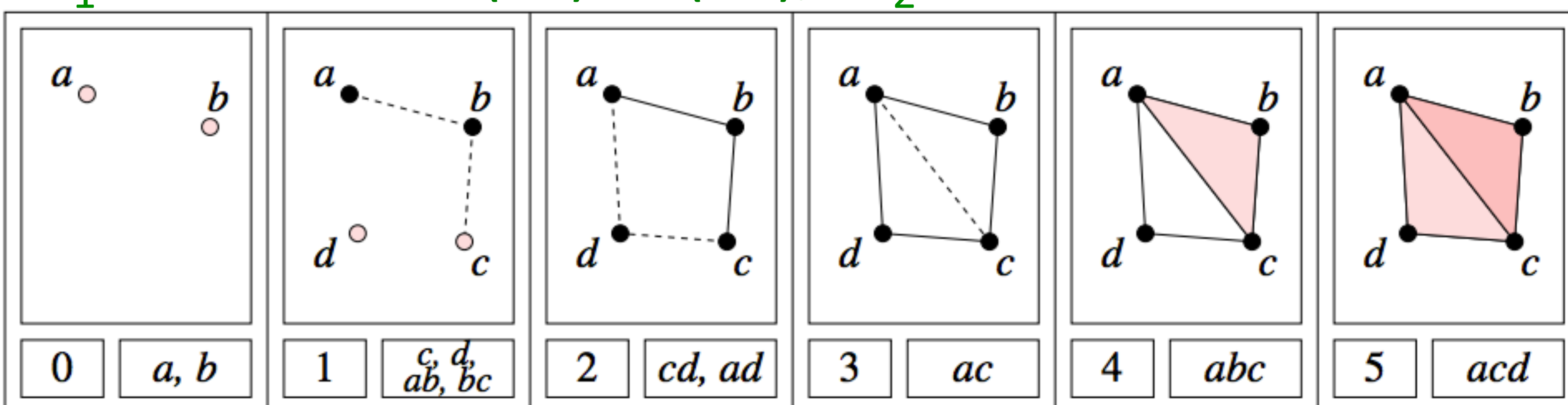
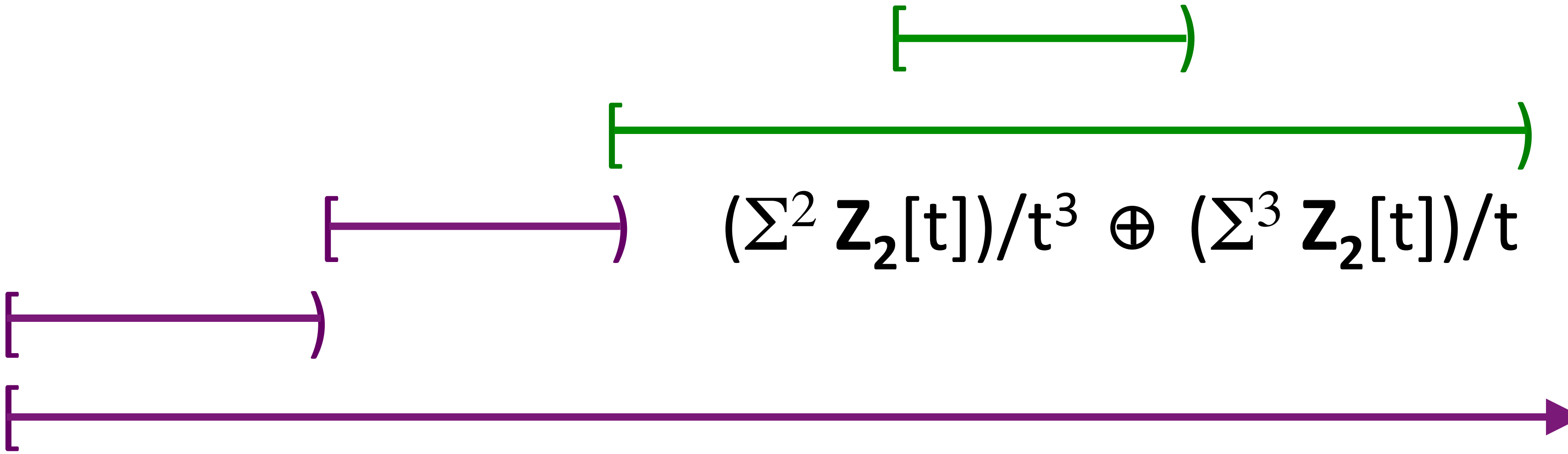


$$z_1 = ad + cd + t(bc) + t(ab), \quad z_2 = ac + t^2 bc + t^2 ab$$



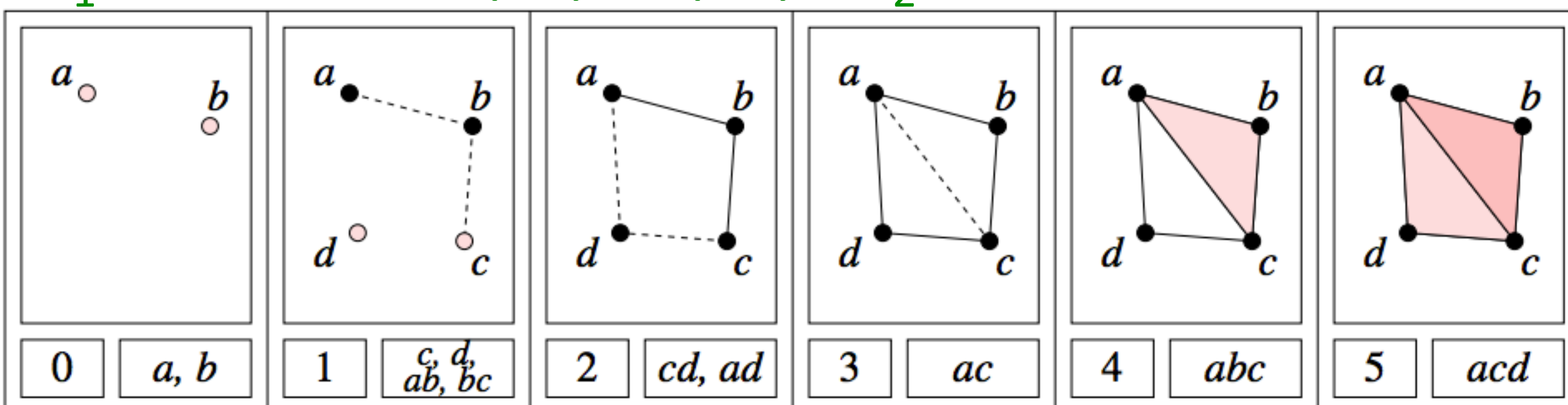
$$H_0 = < a, b, c, d \;:\; tc + td,\; tb + c,\; ta + tb >$$

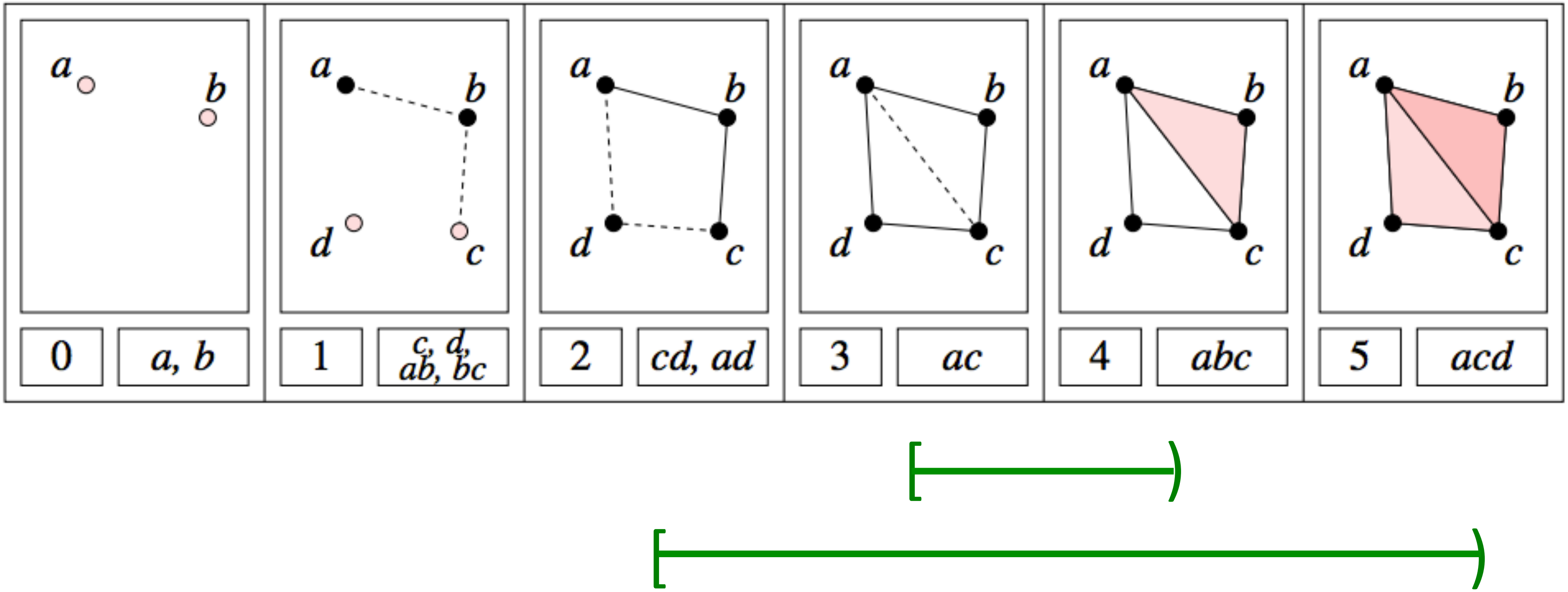
$$H_1 = < z_1, z_2 \;:\; t\,z_2,\; t^3 z_1 + t^2 z_2 >$$



$$(\Sigma^2 \mathbf{Z_2}[t])/t^3 \oplus (\Sigma^3 \mathbf{Z_2}[t])/t$$

$$z_1 = ad + cd + t(bc) + t(ab), \quad z_2 = ac + t^2 bc + t^2 ab$$



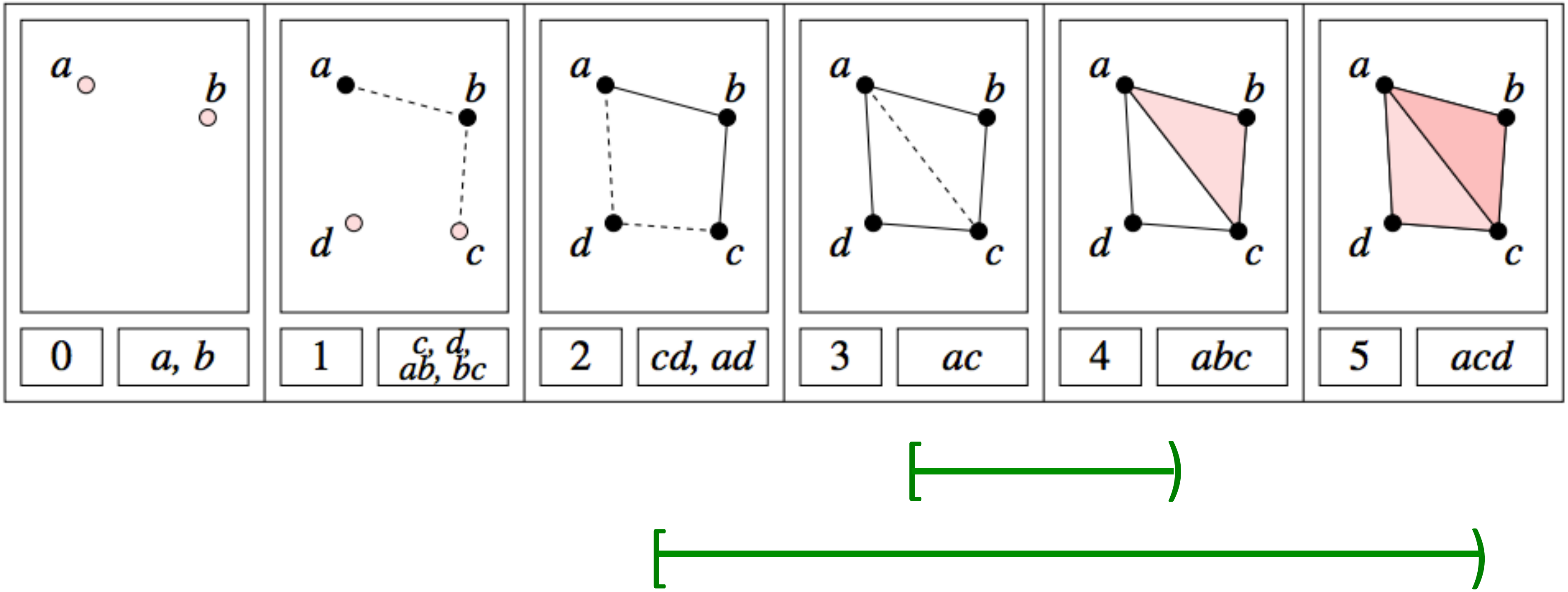


$$H_1 = \langle z_1, z_2 \ : \ t\, z_2,\ t^3 z_1 + t^2 z_2 \rangle$$

$$\mathbf{Z_2}[t] = \{n_0 + n_1 t + n_2 t^2 + n_2 t^3 + n_2 t^4 + \ldots \ : \ n_i \text{ in } \mathbf{Z_2}\}$$

$$z_1: \quad (\Sigma^2\, \mathbf{Z_2}[t])/t^3$$

$$= \{ (0, 0, n_0, n_1, n_2, 0, 0, 0, \ldots) \ : \ n_i \text{ in } \mathbf{Z_2}\}$$



$$H_1 = < z_1, z_2 : t\, z_2, \; t^3 z_1 + t^2 z_2 >$$

$$\mathbf{Z_2}[t] = \{n_0 + n_1 t + n_2 t^2 + n_2 t^3 + n_2 t^4 + \dots : n_i \text{ in } \mathbf{Z_2}\}$$

$$z_2: \quad (\Sigma^3 \mathbf{Z_2}[t])/t$$

$$= \{ (0, 0, 0, n_0, 0, 0, 0, \dots) : n_i \text{ in } \mathbf{Z_2}\}$$

In general when calculating homology over the field **F**

$$H_k = \left(\bigoplus_{i=1}^{n} \Sigma^{\alpha_i} \mathbf{F}[t] \right) \oplus \left(\bigoplus_{i=1}^{n} \Sigma^{\gamma_j} \mathbf{F}[t]/(t^{k_j}) \right)$$

Lecture 9: Visualizing Data via Homology

Friday June 19, 2009

Coffee	
"Visualizing data via homology" image statistics data, range patches, neuroscience	**Gunnar Carlsson** (Stanford University)

http://www.ima.umn.edu/videos/?id=856

http://ima.umn.edu/2008-2009/ND6.15-26.09/activities/Carlsson-Gunnar/imafive-handout4up.pdf

Topological Methods for Large and Complex Data Sets

IMA Workshop on Machine Learning, Minneapolis

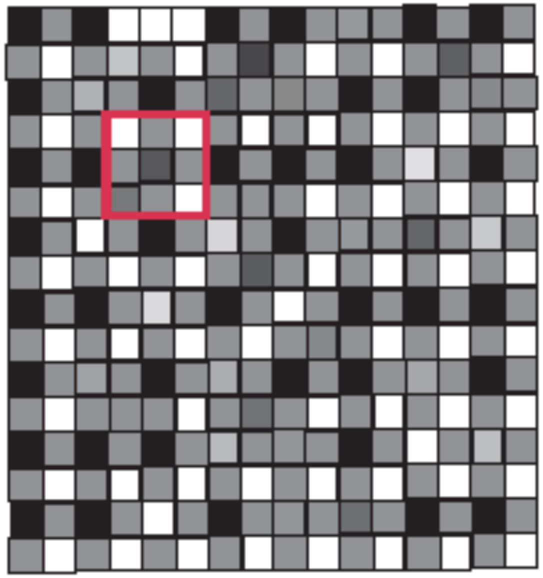
Application to Natural Image Statistics
With V. de Silva, T. Ishkanov, A. Zomorodian

Gunnar Carlsson, Stanford University

March 26, 2012

http://www.ima.umn.edu/videos/?id=1846

http://www.ima.umn.edu/2011-2012/W3.26-30.12/activities/Carlsson-Gunnar/imamachinefinal.pdf



An image taken by black and white digital camera can be viewed as a vector, with one coordinate for each pixel

Each pixel has a “gray scale” value, can be thought of as a real number (in reality, takes one of 255 values)

Typical camera uses tens of thousands of pixels, so images lie in a very high dimensional space, call it pixel space, P