MATH:7450 (22M:305) Topics in Topology: Scientific and Engineering Applications of Algebraic Topology

Sept 13, 2013: Persistent homology II

Fall 2013 course offered through the
University of Iowa Division of Continuing Education

Isabel K. Darcy, Department of Mathematics
Applied Mathematical and Computational Sciences,
University of Iowa

http://www.math.uiowa.edu/~idarcy/AppliedTopology.html

Discrete Comput Geom 28:511–533 (2002)
DOI: 10.1007/s00454-002-2885-2

Discrete & Computational
Geometry

http://link.springer.com/article/10.1007/s00454-002-2885-2

Topological Persistence and Simplification*

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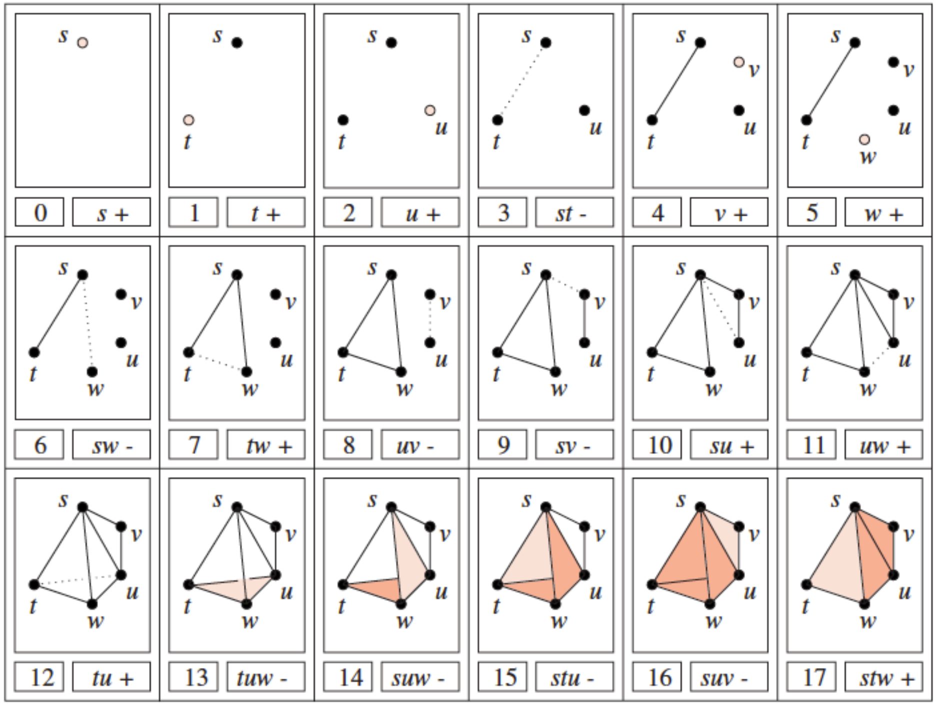
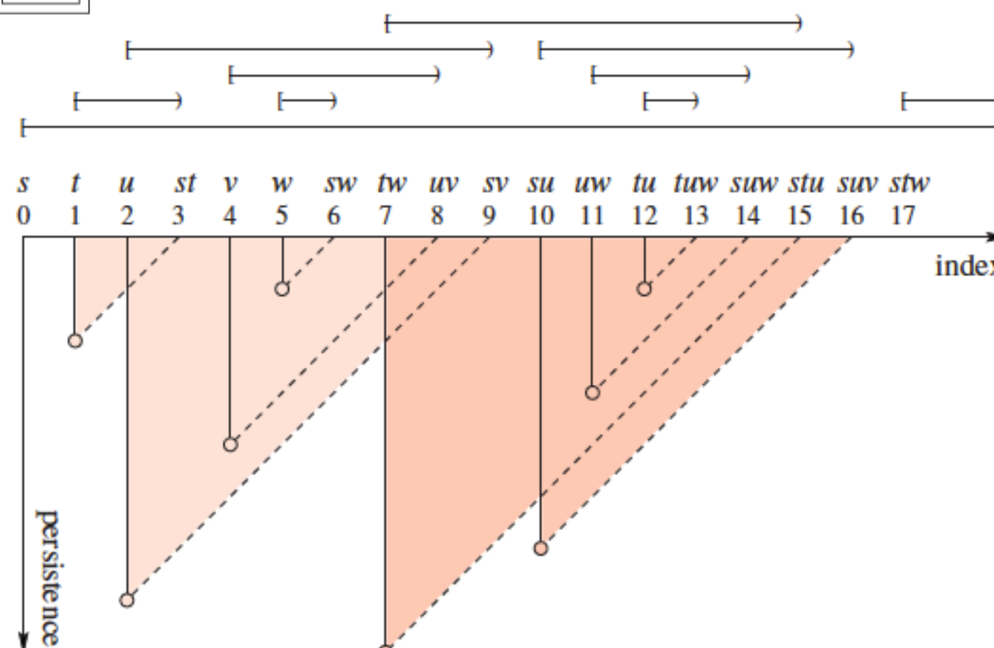
Urbana, IL 61801, USA



A *filtered complex* is an increasing sequence of simplicial complexes:

$C^0 \subset C^1 \subset C^2 \subset \ldots$



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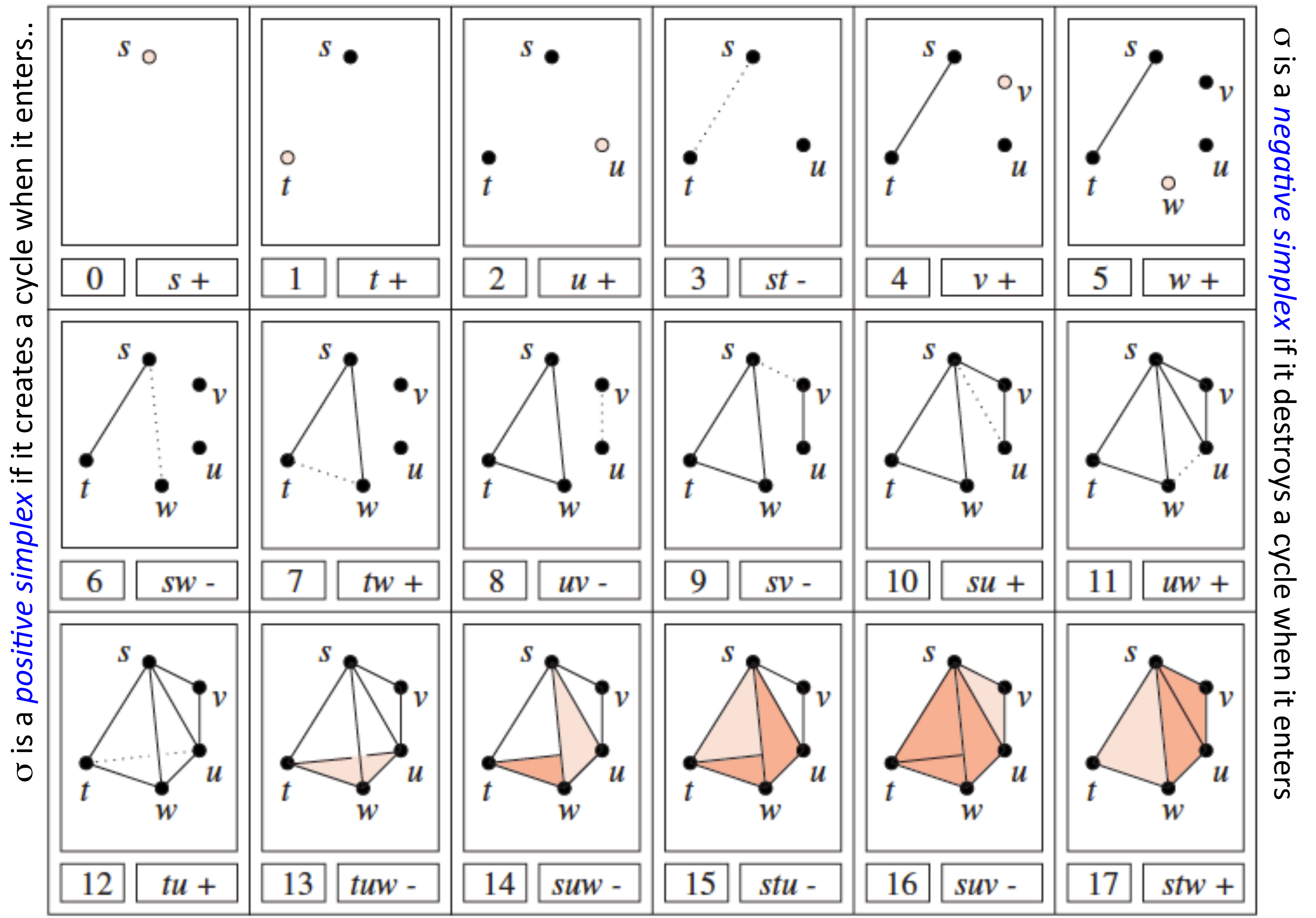

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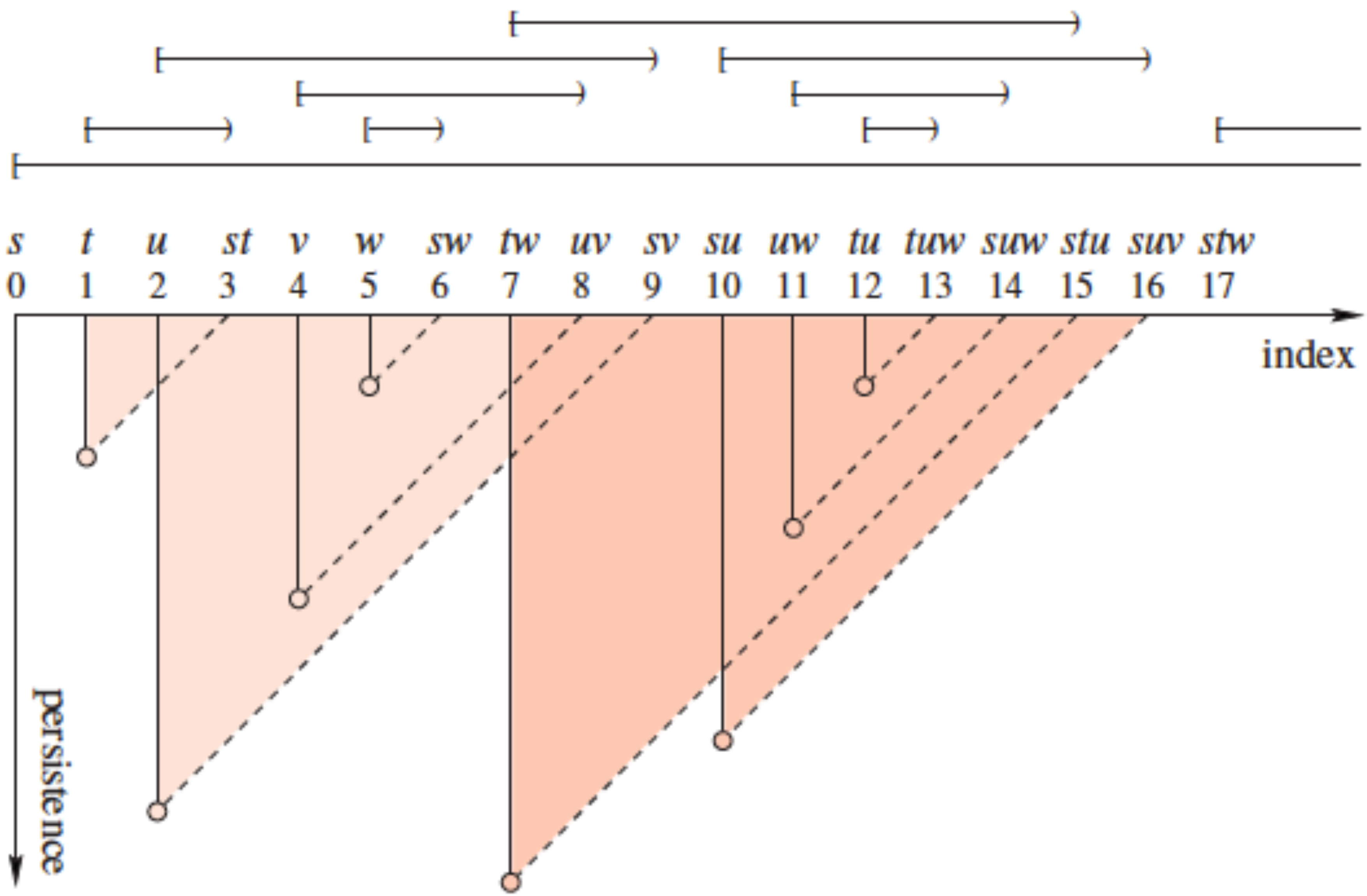

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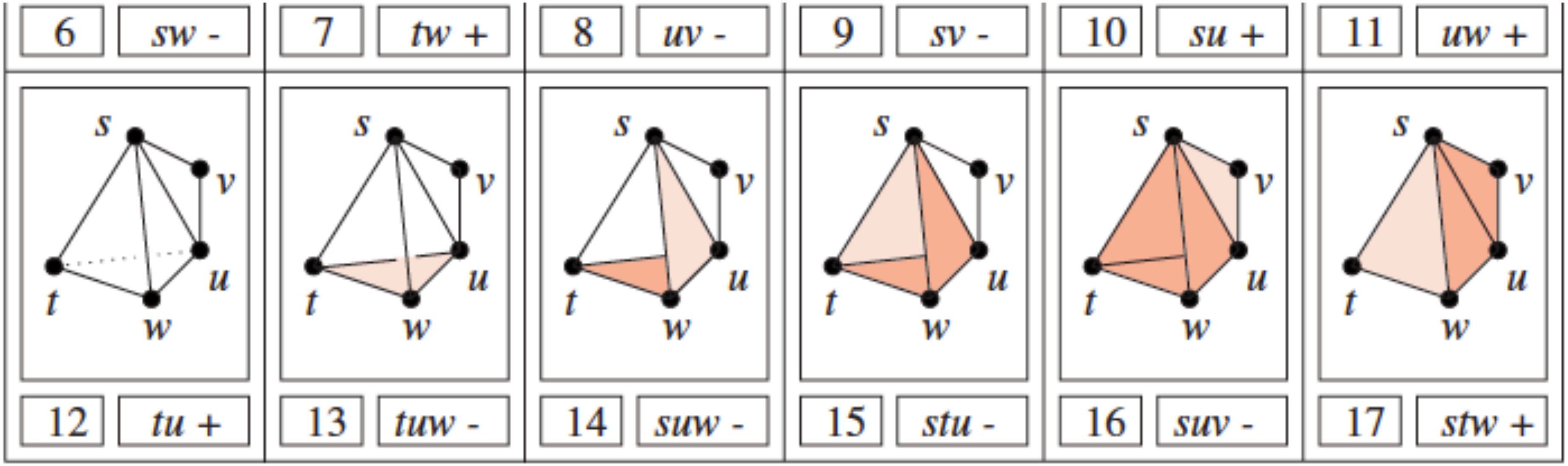
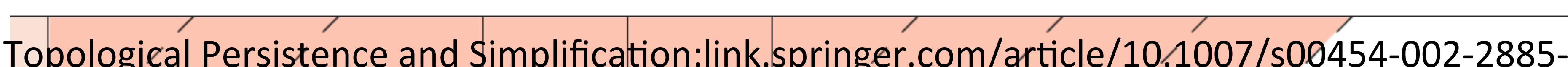


tw	*uv*	*sv*	*su*	*uw*	*tu*	*tuw*	*suw*	*stu*	*suv*	*stw*
7	8	9	10	11	12	13	14	15	16	17



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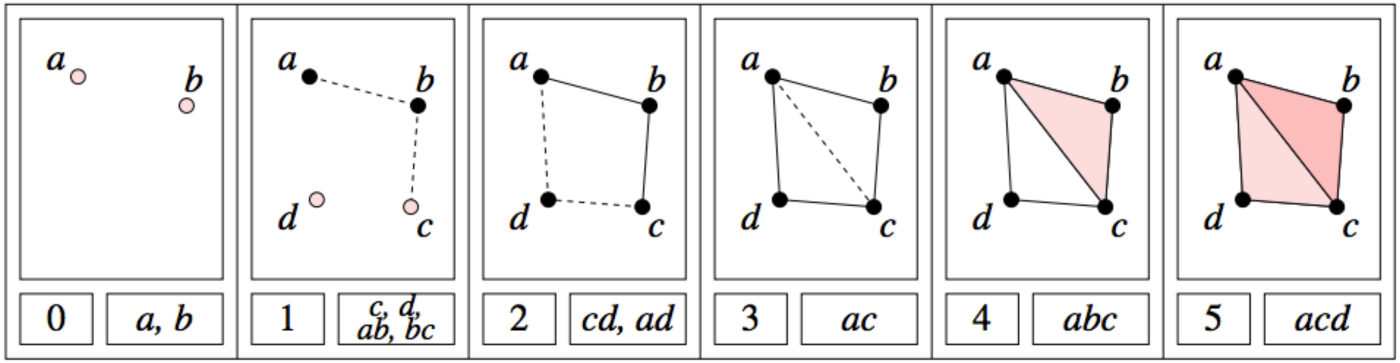

$$
\begin{array}{ccccccc}
C_2^0 & \xrightarrow{f^0} & C_2^1 & \xrightarrow{f^1} & C_2^2 & \xrightarrow{f^2} & \cdots \\
\downarrow \partial_2 & & \downarrow \partial_2 & & \downarrow \partial_2 & & \\
C_1^0 & \xrightarrow{f^0} & C_1^1 & \xrightarrow{f^1} & C_1^2 & \xrightarrow{f^2} & \cdots \\
\downarrow \partial_1 & & \downarrow \partial_1 & & \downarrow \partial_1 & & \\
C_0^0 & \xrightarrow{f^0} & C_0^1 & \xrightarrow{f^1} & C_0^2 & \xrightarrow{f^2} & \cdots
\end{array}
$$

p-persistent k^{th} homology group:

$$H_k^{i,p} = Z_k^i / (B_k^{i+p} \cap Z_k^i)$$

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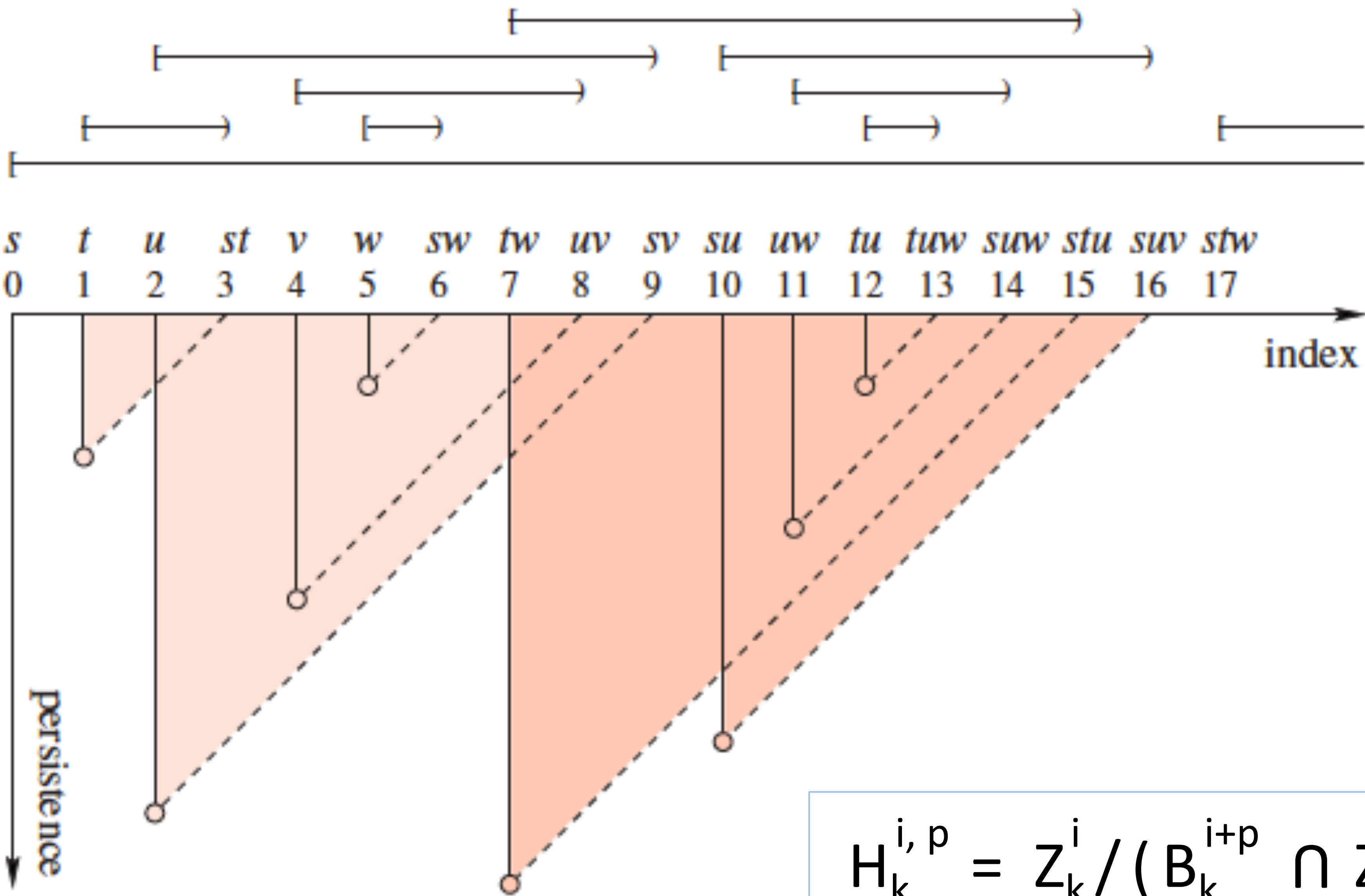

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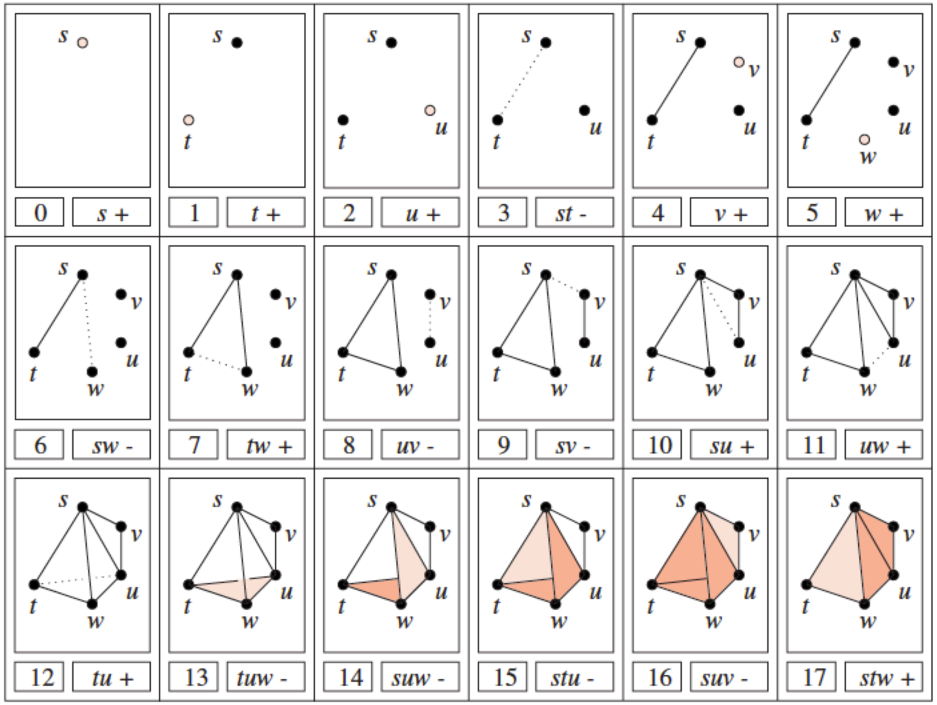

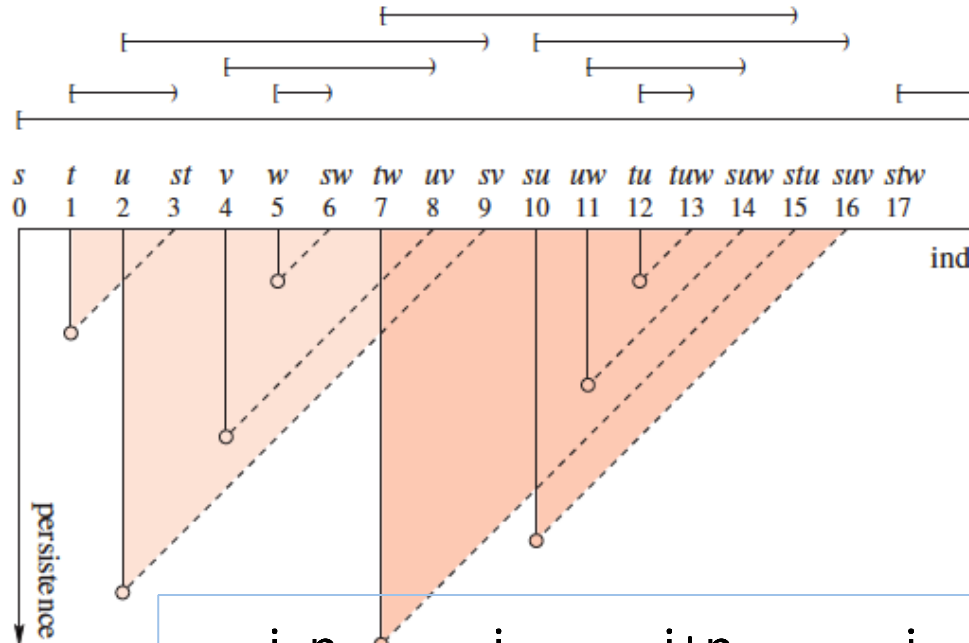

$$H_k^{i,\,p} = Z_k^i / (B_k^{i+p} \cap Z_k^i)$$

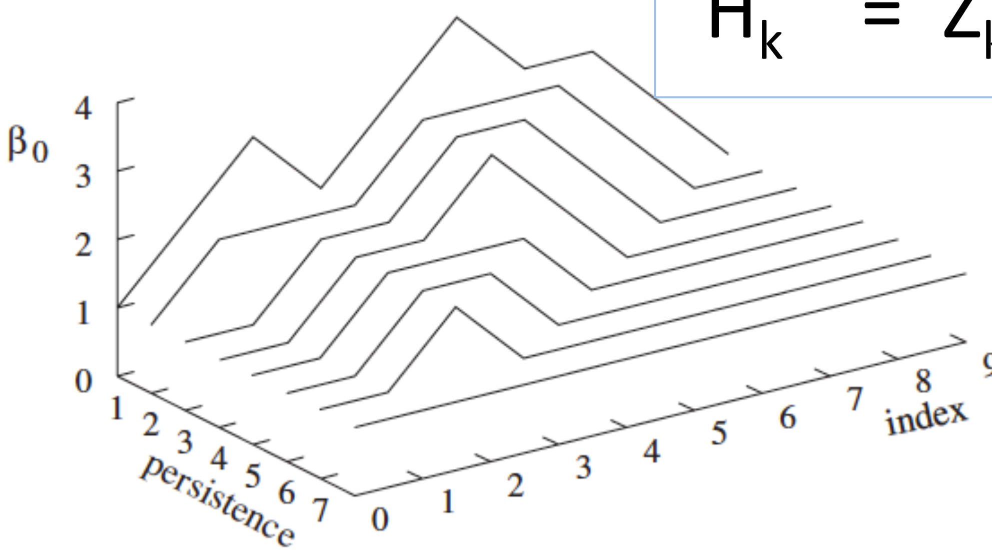

Fig. 11. Persistent 0th Betti numbers of the first ten complexes in the filtration of Fig. 3 and for persistence up to 7.

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Computing Persistent Homology*

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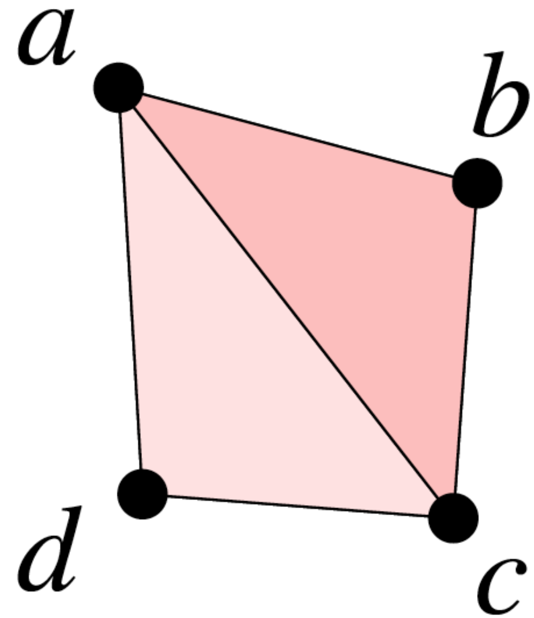
Computing Persistent Homology by Afra Zomorodian, Gunnar Carlsson



$$M_1 = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{ccccc} ab & bc & cd & ad & ac \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

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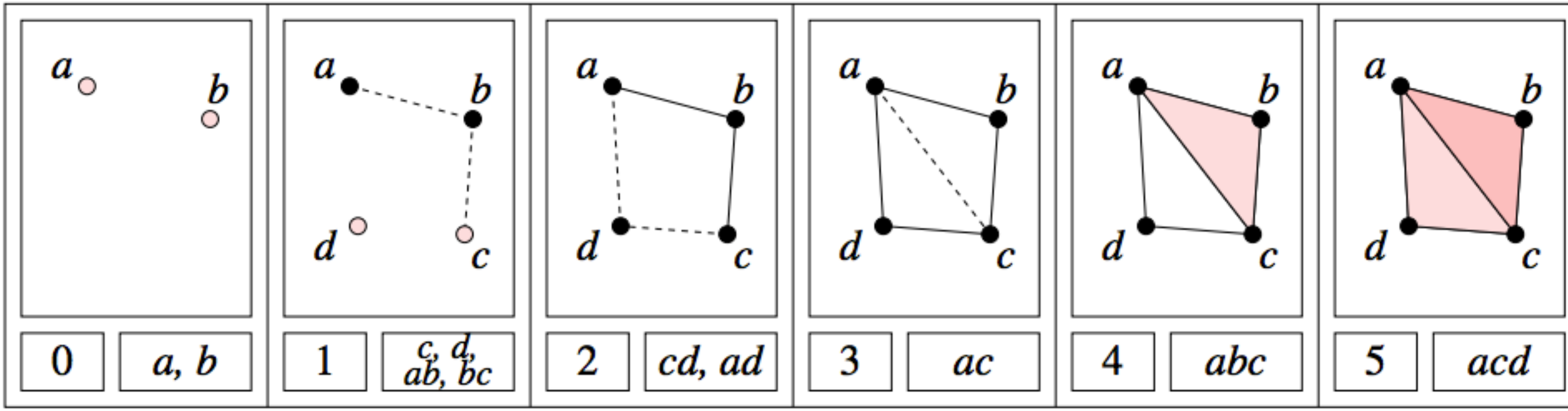

$$M_1 = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{ccccc} ab & bc & cd & ad & ac \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

http://link.springer.com/article/10.1007%2Fs00454-004-1146-y

0	**0**	**1**	**1**	**1**	**1**	**2**	**2**	**3**	**4**	**5**
+	+	+	+	-	-	-	+	+	-	-
a	b	c	d	ab	bc	cd	ad	ac	abc	acd
0	1	2	3	4	5	6	7	8	9	10

$$M_1 = \begin{array}{c} \\ d \\ c \\ b \\ a \end{array} \begin{array}{c} \begin{array}{ccccc} ab & bc & cd & ad & ac \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

0	0	1	1	1	1	2	2	3	4	5
+	+	+	+	-	-	-	+	+	-	-
a	b	c	d	ab	bc	cd	ad	ac	abc	acd
0	1	2	3	4	5	6	7	8	9	10

$$M_1 = \begin{array}{c} \\ d \\ c \\ b \\ a \end{array} \begin{array}{c} \begin{array}{ccccc} ab & bc & cd & ad & ac \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

$$H_k^{i,\,p} = Z_k^i / (B_k^{i+p} \cap Z_k^i)$$

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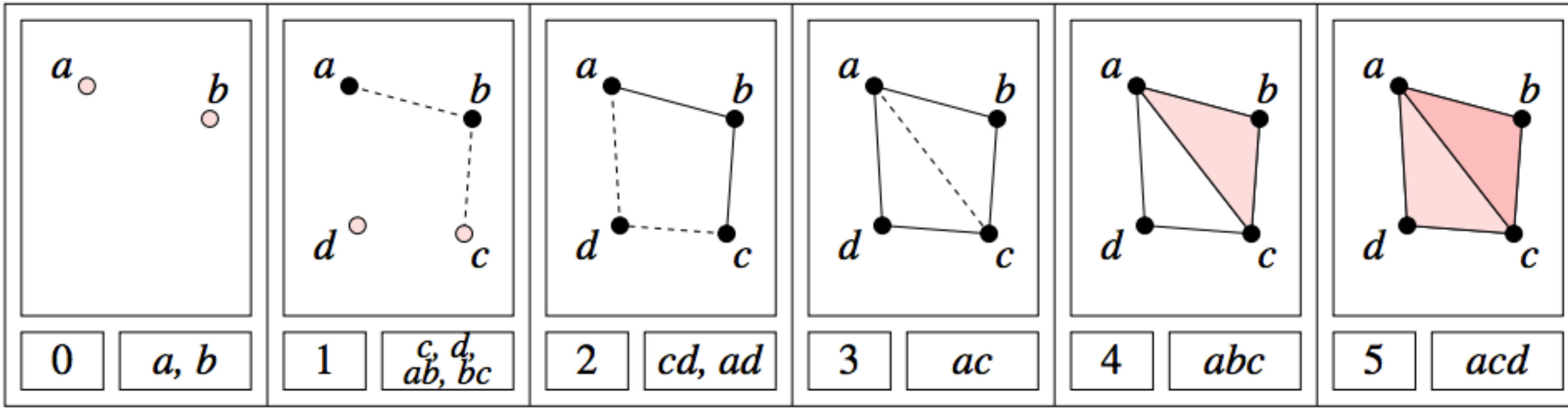

$$M_1 = \left[\begin{array}{c|ccccc} & ab & bc & cd & ad & ac \\ \hline d & 0 & 0 & t & t & 0 \\ c & 0 & 1 & t & 0 & t^2 \\ b & t & t & 0 & 0 & 0 \\ a & t & 0 & 0 & t^2 & t^3 \end{array}\right]$$

$H_k^{i,p} =$

$Z_k^i / (B_k^{i+p} \cap Z_k^i)$

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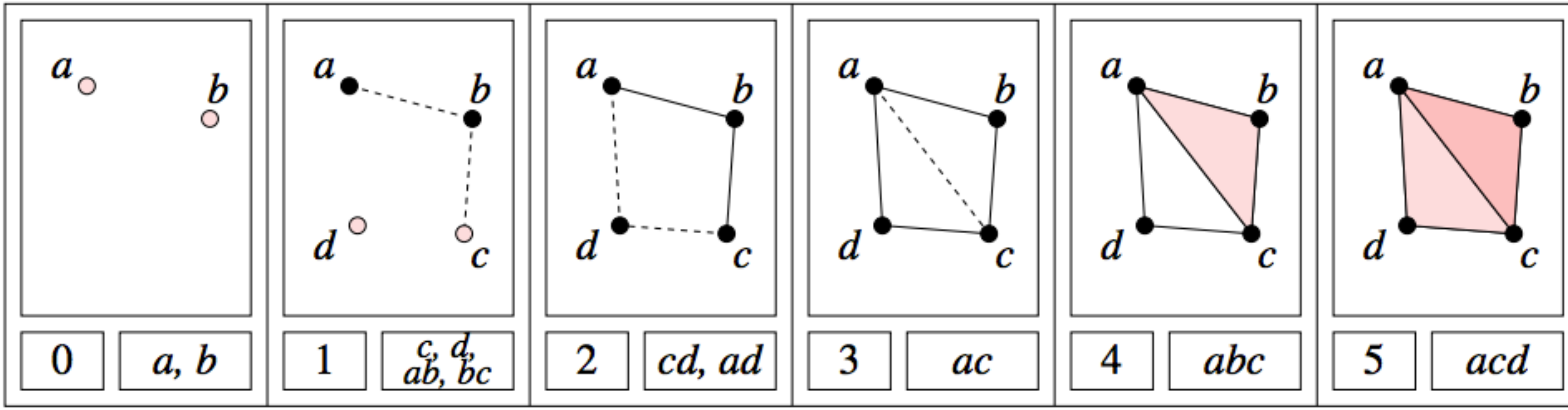

Let σ_i be a k-simplex.

Defn: *degree* of σ_i = deg σ_i

= time when σ_i enters the filtration.

deg a = deg b = 0; deg c = deg d = deg ab = deg bc = 1

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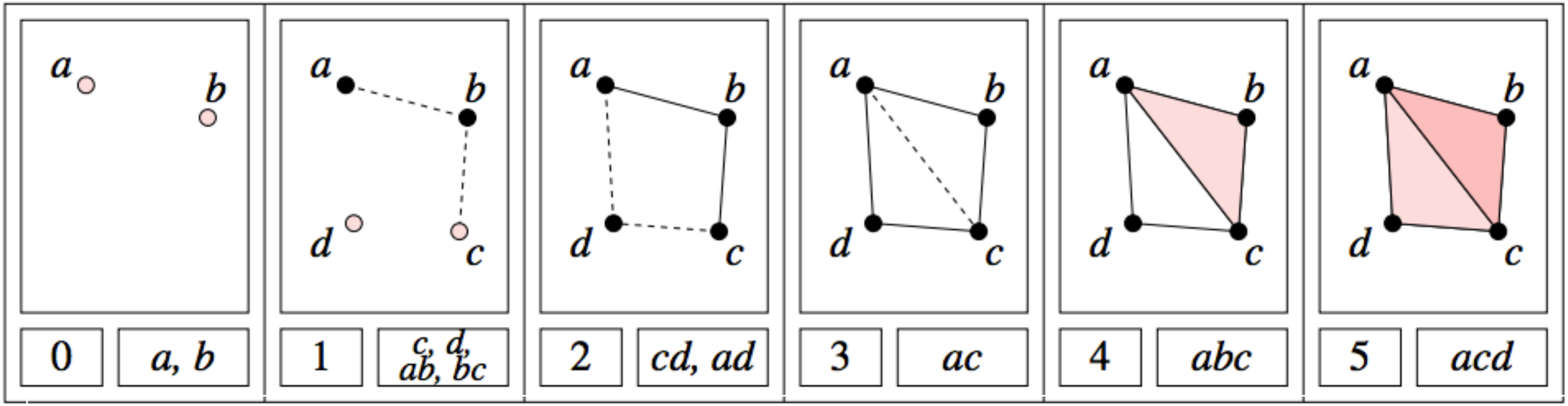

Let σ_i be a k-simplex.

Defn: degree of σ_i = deg σ_i

= time when σ_i enters the filtration.

Defn: deg $t^n \sigma_i$ = n + deg σ_i

A polynomial is *homogeneous* if all terms have the same degree: $t^3a - t^2c + ac$ is in C^3

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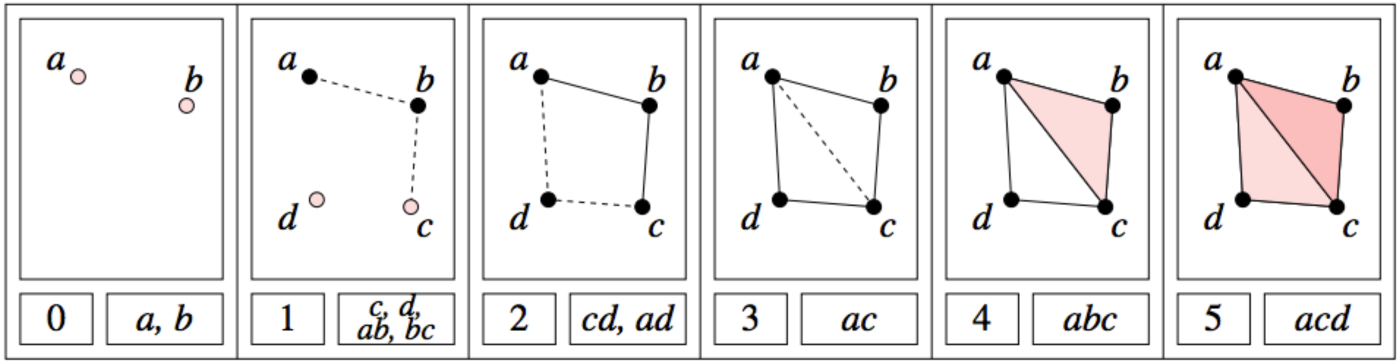

$$
\begin{array}{ccccccc}
\downarrow & & & & & & \\
C_2^0 & \xrightarrow{f^0} & C_2^1 & \xrightarrow{f^1} & C_2^2 & \xrightarrow{f^2} & \cdots \\
\downarrow \partial_2 & & \downarrow \partial_2 & & \downarrow \partial_2 & & \\
C_1^0 & \xrightarrow{f^0} & C_1^1 & \xrightarrow{f^1} & C_1^2 & \xrightarrow{f^2} & \cdots \\
\downarrow \partial_1 & & \downarrow \partial_1 & & \downarrow \partial_1 & & \\
C_0^0 & \xrightarrow{f^0} & C_0^1 & \xrightarrow{f^1} & C_0^2 & \xrightarrow{f^2} & \cdots
\end{array}
$$

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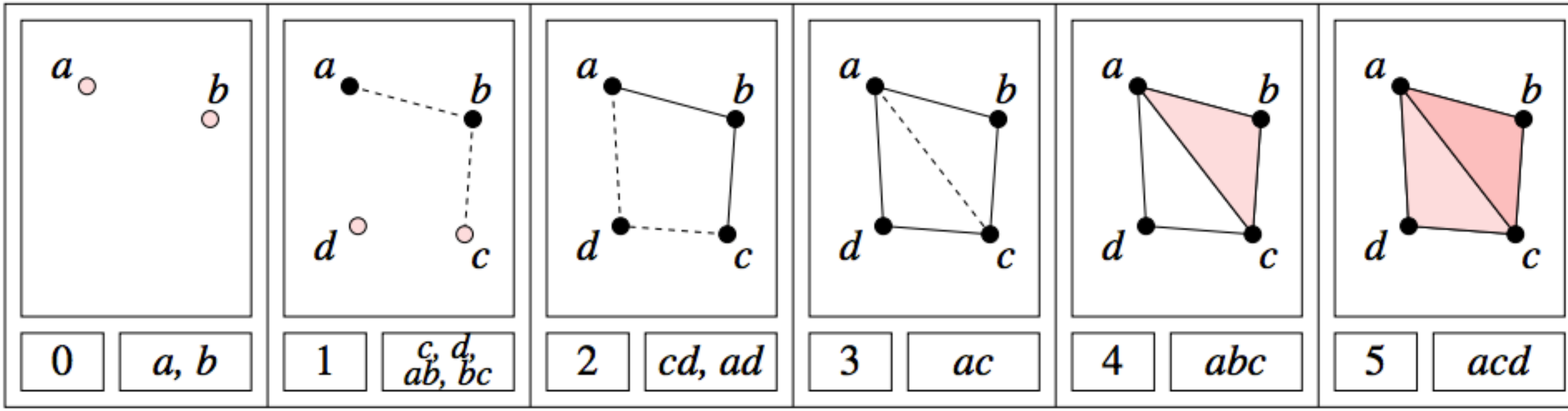

$$M_1 = \left[\begin{array}{c|ccccc} & ab & bc & cd & ad & ac \\ \hline d & 0 & 0 & t & t & 0 \\ c & 0 & 1 & t & 0 & t^2 \\ b & t & t & 0 & 0 & 0 \\ a & t & 0 & 0 & t^2 & t^3 \end{array}\right]$$

$H_k^{i,p} = Z_k^i / (B_k^{i+p} \cap Z_k^i)$

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Let $\{ e_j \}$ be a be a homogeneous basis for C_k

Let $\{ \hat{e}_i \}$ be a be a homogeneous basis for C_{k-1}

$$C_k \xrightarrow{\partial_k} C_{k-1}$$

$$\deg \hat{e}_i + \deg M_k(i, j) = \deg e_j$$

$$H_k^{i,\,p} = Z_k^i / (B_k^{i+p} \cap Z_k^i)$$

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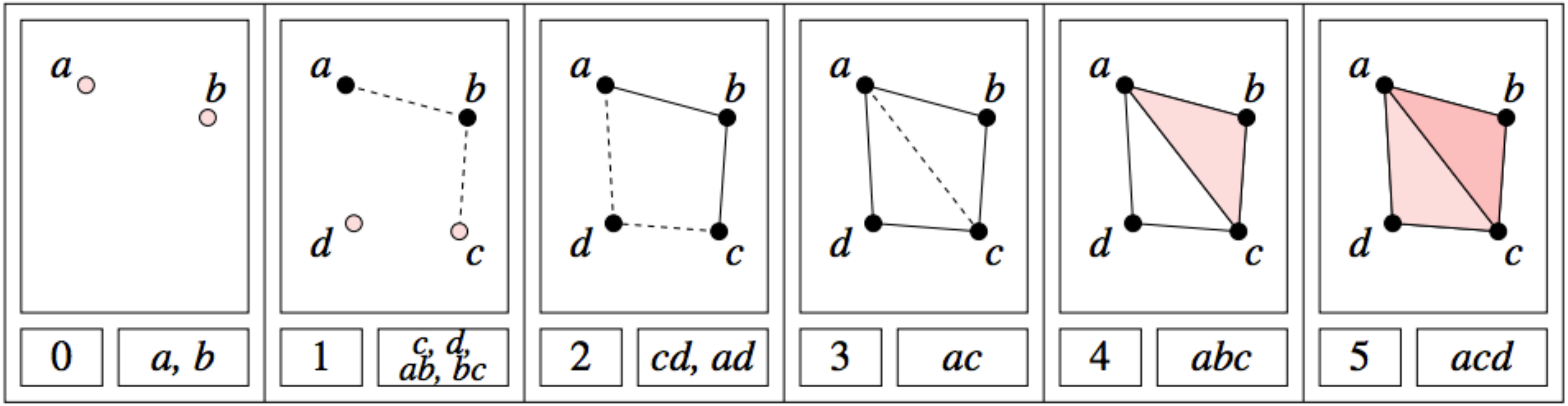

$$M_1 = \left[\begin{array}{c|ccccc} & ab & bc & cd & ad & ac \\ \hline d & 0 & 0 & t & t & 0 \\ c & 0 & 1 & t & 0 & t^2 \\ b & t & t & 0 & 0 & 0 \\ a & t & 0 & 0 & t^2 & t^3 \end{array}\right]$$

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$$\begin{array}{c} \\ d \\ c \\ b \\ a \end{array}\begin{array}{c} \begin{array}{ccccc} ab & bc & cd & ad & ac \end{array} \\ \begin{pmatrix} 0 & 0 & t & t & 0 \\ 0 & 1 & t & 0 & t^2 \\ t & t & 0 & 0 & 0 \\ t & 0 & 0 & t^2 & t^3 \end{pmatrix} \end{array} \qquad \begin{array}{c} \\ d \\ c \\ b \\ a \end{array}\begin{array}{c} \begin{array}{ccccc} cd & bc & ab & ad & ac \end{array} \\ \begin{pmatrix} t & 0 & 0 & t & 0 \\ t & 1 & 0 & 0 & t^2 \\ 0 & t & t & 0 & 0 \\ 0 & 0 & t & t^2 & t^3 \end{pmatrix} \end{array}$$

$$\begin{array}{c} \\ d \\ c \\ b \\ a \end{array}\begin{array}{c} \begin{array}{ccccc} cd & bc & ab & ad+cd & ac \end{array} \\ \begin{pmatrix} t & 0 & 0 & 0 & 0 \\ t & 1 & 0 & t & t^2 \\ 0 & t & t & 0 & 0 \\ 0 & 0 & t & t^2 & t^3 \end{pmatrix} \end{array}$$

$$\begin{array}{c} \\ d \\ c \\ b \\ a \end{array}\begin{array}{c} \begin{array}{ccccc} cd & bc & ab & ad+cd & ac \end{array} \\ \begin{pmatrix} t & 0 & 0 & 0 & 0 \\ t & 1 & 0 & t & t^2 \\ 0 & t & t & 0 & 0 \\ 0 & 0 & t & t^2 & t^3 \end{pmatrix} \end{array}$$

$$\begin{array}{c} \\ d \\ c \\ b \\ a \end{array}\begin{array}{c} \begin{array}{ccccc} cd & bc & ab & ad+cd+t\cdot bc & ac+t^2\cdot bc \end{array} \\ \begin{pmatrix} t & 0 & 0 & 0 & 0 \\ t & 1 & 0 & 0 & 0 \\ 0 & t & t & t^2 & t^3 \\ 0 & 0 & t & t^2 & t^3 \end{pmatrix} \end{array}$$

$$
\begin{array}{c}
 \\ d \\ c \\ b \\ a
\end{array}
\begin{array}{ccccc}
cd & bc & ab & ad + cd + t \cdot bc & ac + t^2 \cdot bc \\
\end{array}
$$

$$
\begin{array}{c|ccccc}
 & cd & bc & ab & ad + cd + t \cdot bc & ac + t^2 \cdot bc \\
\hline
d & t & 0 & 0 & 0 & 0 \\
c & t & 1 & 0 & 0 & 0 \\
b & 0 & t & t & t^2 & t^3 \\
a & 0 & 0 & t & t^2 & t^3
\end{array}
$$

$$
\begin{array}{c|ccccc}
 & cd & bc & ab & ad + cd + t \cdot bc + t \cdot ab & ac + t^2 \cdot bc + t^2 \cdot ab \\
\hline
d & t & 0 & 0 & 0 & 0 \\
c & t & 1 & 0 & 0 & 0 \\
b & 0 & t & t & 0 & 0 \\
a & 0 & 0 & t & 0 & 0
\end{array}
$$

$$\begin{array}{c} \\ d \\ c \\ b \\ a \end{array} \begin{array}{c} \begin{array}{ccccc} cd & bc & ab & ad + cd + t \cdot bc + t \cdot ab & ac + t^2 \cdot bc + t^2 \cdot ab \end{array} \\ \begin{pmatrix} t & 0 & 0 & 0 & 0 \\ t & 1 & 0 & 0 & 0 \\ 0 & t & t & 0 & 0 \\ 0 & 0 & t & 0 & 0 \end{pmatrix} \end{array}$$

B_0 = image of ∂_1 = column space of M_1

= < tc + td, tb + c, ta + tb>

Z_1 = kernel of ∂_1 = null space of M_1

= < ad + cd + t(bc) + t(ab), ac + t^2bc + t^2ab >

$$H_0 = Z_0/B_0 = (\text{kernel of } \partial_0)/(\text{image of } \partial_1)$$

$$= \frac{\text{null space of } M_0}{\text{column space of } M_1}$$

$$= \langle a, b, c, d : tc + td,\ tb + c,\ ta + tb \rangle$$

$$H_0^{i,\,p} = Z_0^i / (B_0^{i+p} \cap Z_0^i)$$

$$H_0^{0,\,0} = \mathbf{Z}_2^2$$

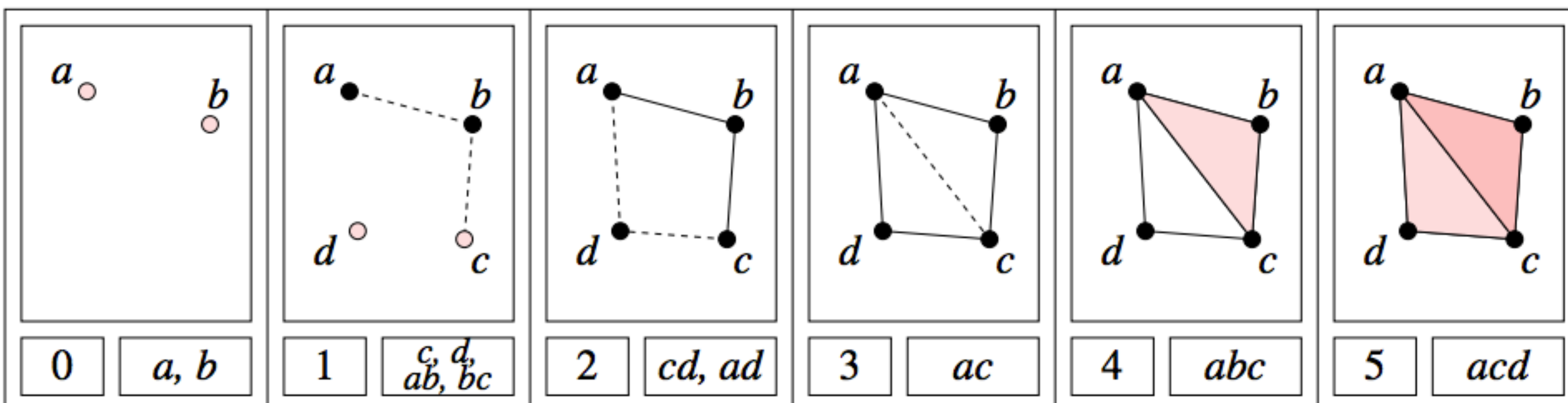
For $p > 0$: $H_0^{0,\,p} = \mathbf{Z}_2$

$$\langle a, b, c, d \ : \ tc + td, tb + c, ta + tb \rangle$$

$$H_0^{i,p} = Z_0^i / (B_0^{i+p} \cap Z_0^i)$$

$$H_0^{0,0} = \mathbf{Z}_2^2$$

For $p > 0$: $H_0^{0,p} = \mathbf{Z}_2$



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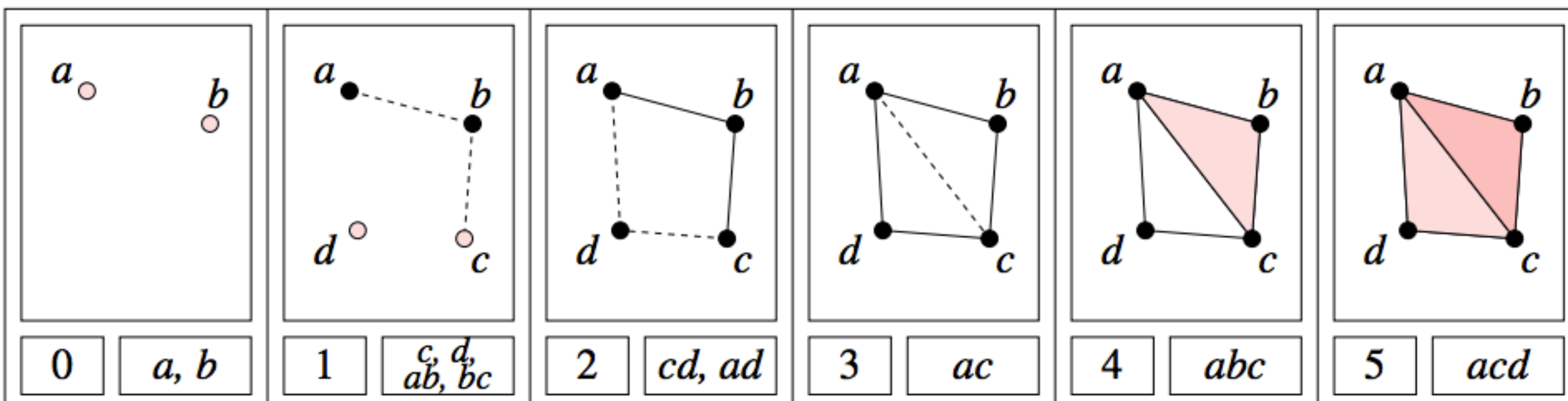
$$\langle a, b, c, d \ : \ tc + td, tb + c, ta + tb \rangle$$

$$H_0^{i,p} = Z_0^i / (B_0^{i+p} \cap Z_0^i)$$

$$H_0^{1,0} = \mathbf{Z}_2^2$$

For $p > 0$: $H_0^{i,p} = \mathbf{Z}_2$

For $i > 1$: $H_0^{i,p} = \mathbf{Z}_2$



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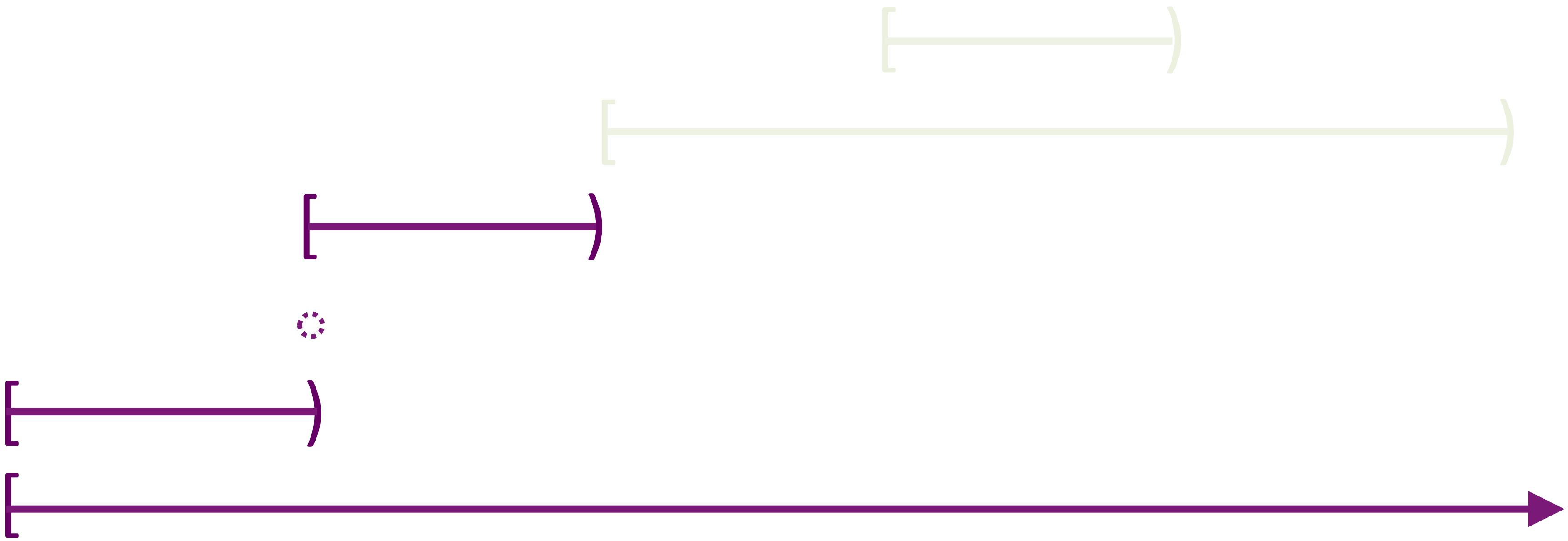
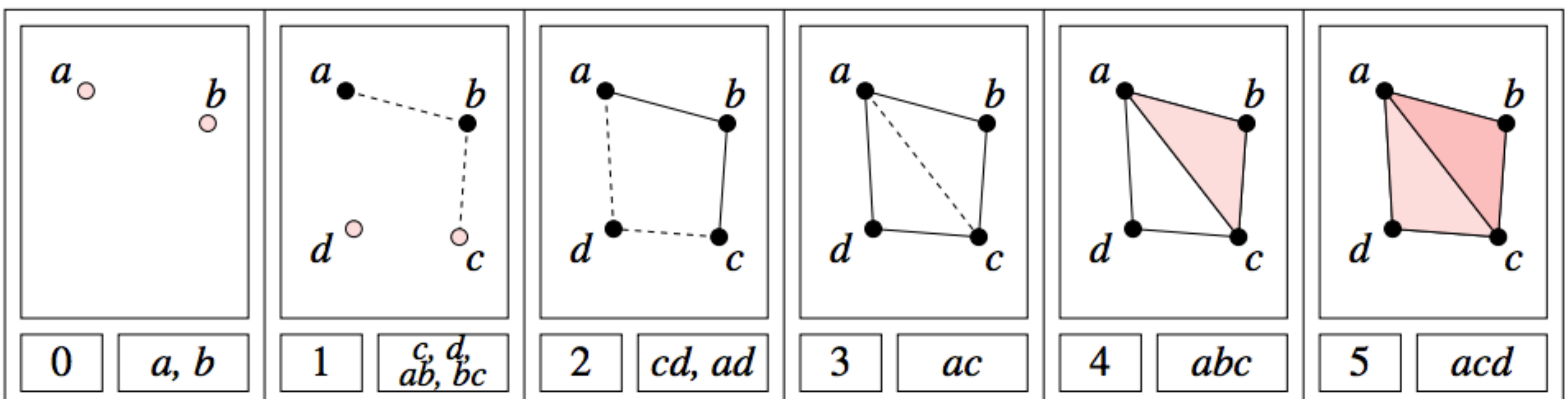
< a, b, c, d : tc + td, tb + c, ta + tb>





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